

An inequality for the number of periods in a word

Daniel Gabric*, Narad Rampersad†, and Jeffrey Shallit*

February 28, 2023

In honor of the 60th birthdays of our colleagues Nelma Moreira and Rogério Reis

Abstract

We prove an inequality for the number of periods in a word x in terms of the length of x and its initial critical exponent. Next, we characterize all periods of the length- n prefix of a characteristic Sturmian word in terms of the lazy Ostrowski representation of n , and use this result to show that our inequality is tight for infinitely many words x . We propose two related measures of periodicity for infinite words. Finally, we also consider special cases where x is overlap-free or squarefree.

1 Introduction

Let x be a finite nonempty word of length n . We say that an integer p , $1 \leq p \leq n$, is a *period* of x if $x[i] = x[i + p]$ for $1 \leq i \leq n - p$. For example, the English word **alfalfa** has periods 3, 6, and 7. A period p is *nontrivial* if $p < n$; the period n is *trivial* and is often ignored. The least period of a word is sometimes called *the period* and is written $\text{per}(x)$. The number of nontrivial periods of a word x is written $\text{np}(x)$. Sometimes the prefix $x[1..p]$ is also called a period; in general, this should cause no confusion.

The *exponent* of a length- n word x is defined to be $\text{exp}(x) = n / \text{per}(x)$. For example, the French word **entente** has exponent $7/3$. The *critical exponent* $\text{ce}(x)$ of a finite or infinite word x is defined to be

$$\text{ce}(x) := \sup\{\text{exp}(p) : p \text{ is a nonempty factor of } x\},$$

and the *initial critical exponent* $\text{ice}(x)$ of a finite or infinite word x is defined to be

$$\text{ice}(x) := \sup\{\text{exp}(p) : p \text{ is a nonempty prefix of } x\}.$$

*School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada; dgabric@uwaterloo.ca; shallit@uwaterloo.ca.

†Department of Math/Stats, University of Winnipeg, 515 Portage Ave., Winnipeg, MB, R3B 2E9 Canada; narad.rampersad@gmail.com.

For example, $\text{ice}(\text{phosphorus}) = 7/4$. This concept was (essentially) introduced by Berthé, Holton, and Zamboni [5].

A word w is a *border* of x if w is both a prefix and a suffix of x . Although overlapping borders are allowed, by convention we generally rule out borders w where $|w| \in \{0, |x|\}$.

There is an obvious relationship between borders and periods: a length- n word x has a nontrivial period t iff it has a border of length $n - t$. For example, the English word *abracadabra* has periods 7, 10, and 11, and borders of length 1 and 4.

A word is *unbordered* if it has no borders and *bordered* otherwise. An unbordered word x has only the trivial period $|x|$. On the other hand, a word of the form a^n , for a a single letter, evidently has the largest possible number of periods; namely, n .

In this note we prove an inequality that gives an upper bound for $\text{nnp}(x)$, the number of nontrivial periods of (and hence, the number of borders in) a word x . Roughly speaking, this inequality says that, in order for a word to have many periods, it must either be very long, or have a large initial critical exponent. We also prove that our inequality is tight, up to an additive constant. To do so, in Section 3 we characterize all periods of the length- n prefix of a characteristic Sturmian word in terms of the lazy Ostrowski representation of n . In Section 5, we propose two related measures of periodicity for infinite words, and we compute these measure for some famous words. Finally, in the last two sections, we consider the shortest binary overlap-free (resp., ternary squarefree) words having n periods.

2 The period inequality

The original motivation for this paper was to answer the following question: when does a word have many periods?

Obviously, one way a word can have many periods is if it is periodic: for example, 0^n has n periods. More generally, a word with high exponent has many periods.

On the other hand, the word $0^n 1^{n^2} 0^n$ has many periods, but very small exponent—namely, $(n^2 + 2n)/(n^2 + n) \approx 1 + 1/n$. So exponent alone can't be the whole story.

One might be tempted to consider the critical exponent $\text{ce}(x)$. However, a word such as $01^n 0$ has only one period, but has high critical exponent.

Instead we will consider the initial critical exponent $\text{ice}(x)$ and relate it to $\text{nnp}(x)$, the number of nontrivial periods.

Theorem 1. *Let x be a bordered word of length $n \geq 1$. Let $\beta = \text{ice}(x)$. Then*

$$\text{nnp}(x) \leq \frac{\beta}{2} + 1 + \frac{\ln(n/2)}{\ln(\beta/(\beta - 1))}. \quad (1)$$

Proof. We break the bound up into two pieces, by considering the periods of size $\leq n/2$ and $> n/2$. We call these the *short* and *long* periods.

Let $p = \text{per}(x)$, the shortest period of x . If p is short, then x has short periods $p, 2p, 3p, \dots, \lfloor n/(2p) \rfloor p$. Clearly $\text{ice}(x) \geq n/p$, so we get at most $\beta/2$ short periods from

this list. To see that there are no other short periods, let q be some short period not on this list. Then $p < q \leq n/2$ by assumption. By the Fine-Wilf theorem [12], if a word of length n has two periods p, q with $n \geq p + q - \gcd(p, q)$, then it also has period $\gcd(p, q)$. Since $\gcd(p, q) \leq p$, either $\gcd(p, q) < p$, which is a contradiction, or $\gcd(p, q) = p$, which means q is a multiple of p , another contradiction.

Next, let's consider the long periods or, alternatively, the short borders (those of length $< n/2$). Suppose x has borders y, z of length q and r respectively, with $q < r < n/2$. Then $x = yy'y = zz'z$ for words y' and z' . Hence $z = yt = t'y$ for some nonempty words t and t' . Then by the Lyndon-Schützenberger theorem (see, e.g., [16]) we know there exist words u, v with u nonempty, and an integer $d \geq 0$, such that $t' = uv$, $t = vu$, and $y = (uv)^d u$. Hence x has the prefix $z = yt = (uv)^{d+1}u$, which means $\beta = \text{ice}(x) \geq |z|/|uv| = r/(r - q)$.

Now the inequality $r/(r - q) \leq \beta$ is equivalent to $r/q \geq \beta/(\beta - 1)$. Thus if

$$b_1 < b_2 < \dots < b_t$$

are the lengths of all the short borders of x , by above we have

$$\begin{aligned} b_1 &\geq 1 \\ b_2 &\geq (\beta/(\beta - 1))b_1 \geq \beta/(\beta - 1), \end{aligned}$$

and so forth, and hence $b_t \geq (\beta/(\beta - 1))^{t-1}$. All these borders are of length at most $n/2$, so $n/2 > b_t \geq (\beta/(\beta - 1))^{t-1}$. Hence

$$t \leq 1 + \frac{\ln(n/2)}{\ln(\beta/(\beta - 1))},$$

and the result follows. □

It is also possible to simplify the statement of the bound (1), at the cost of being less precise.

Corollary 2. *Let x be a word of length $n \geq 1$, and let $\beta = \text{ice}(x)$. Then*

- (a) $\text{nnp}(x) \leq \frac{\beta}{2} + 1 + (\beta - \frac{1}{2}) \ln(n/2)$;
- (b) $\text{nnp}(x) \leq C\beta \ln n$, where $C = 3/(2 \ln 2) \doteq 2.164$.

Proof. (a) Start with (1). By computing Taylor series around $z = \infty$, we see that

$$\ln(z/(z - 1)) = \sum_{n \geq 1} \frac{1}{nz^n} < \sum_{n \geq 1} \frac{1}{2^{n-1}z^n} = \frac{1}{z - 1/2}$$

for $z > 1$. Hence if $\beta > 1$ then

$$\frac{1}{\ln(\beta/(\beta - 1))} \leq \beta - \frac{1}{2}.$$

If $\beta = 1$, then x is unbordered. The left-hand side of (a) is then 0, while the right-hand side is at least $3/2 + (1/2) \ln(1/2) \geq 1$.

(b) If $n = 1$ then the desired inequality follows trivially.

Otherwise assume $n \geq 2$. It is easy to check that

$$1 + \frac{1}{2} \ln 2 = \left(\ln 2 - \frac{1}{2} \right) + \frac{1}{2} \ln 2 + (C - 1) \ln 2$$

where $C = 3/(2 \ln 2)$. Thus

$$1 + \frac{1}{2} \ln 2 \leq \left(\ln 2 - \frac{1}{2} \right) \beta + \frac{1}{2} \ln n + (C - 1) \beta \ln n,$$

since $n \geq 2$ and $\beta \geq 1$. Now add $\beta \ln n$ to both sides and rearrange to get

$$\frac{\beta}{2} + 1 + \left(\beta - \frac{1}{2} \right) \ln(n/2) \leq C \beta \ln n,$$

which by (a) gives the desired result. □

It is natural to wonder how tight the bound (1) is for a “typical” word of length n . The following two results imply that the expected value of the left-hand side of (1) is $O(1)$, while the expected value of the right-hand side is $\Theta(\ln n)$. Our inequality, therefore, implies nothing useful about the “typical” word.

Theorem 3. *Let $k \geq 2$. Over a k -letter alphabet, the expected number of borders (or the number of nontrivial periods) of a length- n word is $k^{-1} + k^{-2} + \dots + k^{1-n} \leq \frac{1}{k-1}$.*

Proof. By the linearity of expectation, the expected number of borders is the sum, from $i = 1$ to $n - 1$, of the expected value of the indicator random variable B_i taking the value 1 if there is a border of length i , and 0 otherwise. Once the left border of length i is chosen arbitrarily, the i bits of the right border are fixed, and so there are $n - i$ free choices of symbols. This means that $E[B_i] = k^{n-i}/k^n = k^{-i}$. □

Theorem 4. *The expected value of $\text{ice}(x)$, for finite or infinite words x , is $\Theta(1)$.*

Proof. Let’s count the fraction H_j of words having at least a j ’th power prefix. Count the number of words having a j ’th power prefix with period 1, 2, 3, etc. This double counts, but shows that $H_j \leq k^{1-j} + k^{2(1-j)} + \dots = 1/(k^{j-1} - 1)$ for $j \geq 2$. Clearly $H_1 = 1$.

Then $H_{j-1} - H_j$ is the fraction of words having a $(j - 1)$ th power prefix but no j th power prefix. These words will have an ice at most j . So the expected value of ice is bounded above by

$$\begin{aligned} 2(H_1 - H_2) + 3(H_2 - H_3) + 4(H_3 - H_4) + \dots &= 2H_1 + H_2 + H_3 + H_4 + \dots \\ &= 2 + H_2 + H_3 + H_4 + \dots \\ &= 2 + \sum_{j \geq 2} 1/(k^{j-1} - 1) \\ &= 2 + \sum_{j \geq 1} 1/(k^j - 1). \end{aligned}$$

□

3 Periods of prefixes of characteristic Sturmian words

In this section we take a brief digression to completely characterize the periods of the length- n prefix of the characteristic Sturmian word with slope α . This characterization is based on a remarkable connection between these periods and the so-called “lazy Ostrowski” representation of n . Theorem 6 below implies that all the periods of a length- n prefix of a Sturmian characteristic word can be read off directly from the lazy Ostrowski representation of n .

We start by recalling the Ostrowski numeration system. Let $0 < \alpha < 1$ be an irrational real number with continued fraction expansion $[0, a_1, a_2, \dots]$. Define p_i/q_i to be the i 'th convergent to this continued fraction, so that $p_0 = 0$, $q_0 = 1$, and $[0, a_1, a_2, \dots, a_i] = p_i/q_i$ for $i \geq 1$. In the (ordinary) Ostrowski numeration system, we write every positive integer in the form

$$n = \sum_{0 \leq i \leq t} d_i q_i, \quad (2)$$

where $d_t > 0$ and the d_i have to obey three conditions:

- (a) $0 \leq d_0 < a_1$;
- (b) $0 \leq d_i \leq a_{i+1}$ for $i \geq 1$;
- (c) For $i \geq 1$, if $d_i = a_{i+1}$ then $d_{i-1} = 0$.

See, for example, [1, §3.9].

The *lazy Ostrowski representation* is again defined through the sum (2), but with slightly different conditions:

- (d) $0 \leq d_0 < a_1$;
- (e) $0 \leq d_i \leq a_{i+1}$ for $i \geq 1$;
- (f) For $i \geq 2$, if $d_i = 0$, then $d_{i-1} = a_i$;
- (g) If $d_1 = 0$, then $d_0 = a_1 - 1$.

See, for example, [11, §5]. By convention, the Ostrowski representation is written as a finite word $d_t d_{t-1} \cdots d_1 d_0$, starting with the most significant digit.

Next, we recall the definition of the characteristic Sturmian infinite word $\mathbf{x}_\alpha = x_1 x_2 x_3 \cdots$. It is defined by

$$x_i = \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor$$

for $i \geq 1$. For more about Sturmian words, see [4, 19, 3].

Example 5. Take $\alpha = \sqrt{2} - 1 = [0, 2, 2, 2, \dots]$. Then $q_0 = 1$, $q_1 = 2$, $q_2 = 5$, $q_3 = 12$. The first few ordinary and lazy Ostrowski representations are given in the table below.

n	ordinary Ostrowski	lazy Ostrowski	n	ordinary Ostrowski	lazy Ostrowski
1	1	1	15	1011	221
2	10	10	16	1020	1020
3	11	11	17	1100	1021
4	20	20	18	1101	1101
5	100	21	19	1110	1110
6	101	101	20	1111	1111
7	110	110	21	1120	1120
8	111	111	22	1200	1121
9	120	120	23	1201	1201
10	200	121	24	2000	1210
11	201	201	25	2001	1211
12	1000	210	26	2010	1220
13	1001	211	27	2011	1221
14	1010	220	28	2020	2020

In what follows, fix a suitable α . Let Y_n for $n \geq 1$ be the prefix of \mathbf{x}_α of length n , and define $X_n := Y_{q_n}$. Let $\text{PER}(n)$ denote the set of all periods of Y_n (including the trivial period n). Then we have the following result, which gives a complete characterization of the periods of Y_n . It can be viewed as a generalization of a 2009 theorem of Currie and Saari [9, Corollary 8], which obtained the least period of X_n .

Theorem 6.

- (a) *The number of periods of Y_n (including the trivial period n) is equal to the sum of the digits in the lazy Ostrowski representation of n .*
- (b) *Suppose the lazy Ostrowski representation of n is $\sum_{0 \leq i \leq t} d_i q_i$. Define*

$$A(n) = \left\{ eq_j + \sum_{j < i \leq t} d_i q_i : 1 \leq e \leq d_j \text{ and } 0 \leq j \leq t \right\}.$$

Then $\text{PER}(n) = A(n)$.

Part (a) follows immediately from part (b), so it suffices to prove (b) alone. We need some preliminary lemmas.

Lemma 7. *The lazy Ostrowski representation of n has length $t + 1$ if and only if*

$$q_t + q_{t-1} - 1 \leq n \leq q_{t+1} + q_t - 2.$$

Proof. The largest integer N represented by a lazy Ostrowski representation of length $t + 1$ is the one where the coefficient of each q_i takes the maximum possible values allowed by conditions (d) and (e) above; namely $N = a_1 - 1 + \sum_{1 \leq i \leq t} a_{i+1} q_i$. Suppose t is even; an analogous proof works for the case of t odd. Then

$$\begin{aligned} q_{t+1} &= a_{t+1} q_t + q_{t-1} \\ q_{t-1} &= a_{t-1} q_{t-2} + q_{t-3} \\ &\vdots \\ q_1 &= a_1 q_0 + 0, \end{aligned}$$

which, by adding up and cancelling, gives

$$q_{t+1} = a_{t+1} q_t + a_{t-1} q_{t-2} + \cdots + a_1 q_0. \quad (3)$$

Similarly

$$\begin{aligned} q_t &= a_t q_{t-1} + q_{t-2} \\ q_{t-2} &= a_{t-2} q_{t-3} + q_{t-4} \\ &\vdots \\ q_2 &= a_2 q_1 + q_0, \end{aligned}$$

which, by adding up and cancelling, gives

$$q_t = a_t q_{t-1} + a_{t-2} q_{t-3} + \cdots + a_2 q_1 + q_0. \quad (4)$$

Adding Eqs. (3) and (4) gives $q_t + q_{t+1} = 1 + a_1 q_0 + \sum_{1 \leq i \leq t} a_{i+1} q_i$, and hence $N = q_t + q_{t+1} - 2$, as desired. \square

Lemma 8. *We have $A(n) \subseteq \text{PER}(n)$.*

Proof. Frid [13] defined two kinds of representations in the Ostrowski system. A representation $n = \sum_{0 \leq i \leq t} d_i q_i$ is *legal* if $0 \leq d_i \leq a_{i+1}$. A representation $n = \sum_{0 \leq i \leq t} d_i q_i$ is *valid* if $Y_n = X_t^{d_t} \cdots X_0^{d_0}$. She proved [13, Corollary 1, p. 205] that every legal representation is valid. Since the lazy Ostrowski representation is legal [11, Thm. 47], it follows that if $n = \sum_{0 \leq i \leq t} d_i q_i$ is the lazy Ostrowski representation of n , then $Y_n = X_t^{d_t} \cdots X_0^{d_0}$.

We now argue that each of the following

$$\begin{aligned} &X_t, X_t^2, \dots, X_t^{d_t}, \\ &X_t^{d_t} X_{t-1}, X_t^{d_t} X_{t-1}^2, \dots, X_t^{d_t} X_{t-1}^{d_{t-1}}, \\ &\dots, \\ &X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_1^{d_1} X_0, X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_1^{d_1} X_0^2, \dots, X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_1^{d_1} X_0^{d_0}. \end{aligned} \quad (5)$$

are periods of Y_n .

To see this, we use an argument suggested by a referee, which is a considerable simplification of our original proof. Let U be a word in the above list. Then $Y_n = UV$ for some word V . But since any suffix of a lazy Ostrowski representation is valid, it must be legal, and so by Frid's result, we have $V = Y_{n'}$ for a suitable $n' \leq n$. However, $Y_{n'}$ is a prefix of Y_n . Hence Y_n is a prefix of UY_n , and hence U is a period of Y_n . \square

Before proving the next lemma, we need to recall two results from the theory of Sturmian words. The first is the ‘‘almost commutative’’ property of the X_n (see [1, Theorem 9.1.11]): If ab are the last two symbols of $X_n X_{n-1}$, then $X_n X_{n-1}(ab)^{-1} = X_{n-1} X_n (ba)^{-1}$. We also introduce the central words: a word w is *central* if it has two co-prime periods p and q and $|w| = p + q - 2$. We need the following result:

Proposition 9. [7, Proposition 1] *A word w is central if and only if w is either a power of a single letter or it satisfies the equation*

$$w = w_1 a b w_2 = w_2 b a w_1$$

for some words w_1 and w_2 . Furthermore, in the latter case, the words w_1 and w_2 are central, the quantities $p = |w_1| + 2$ and $q = |w_2| + 2$ are co-prime periods of w , and $\min\{p, q\}$ is the minimal period of w .

One last fact that we recall before introducing the next lemma is that q_i and q_{i+1} are always co-prime. This is a well-known property of the q_i from the theory of continued fractions: it follows, for instance, from the identity $p_i q_{i+1} - q_i p_{i+1} = \pm 1$.

Lemma 10. *If $q_t + q_{t-1} - 1 \leq n \leq q_{t+1} + q_t - 2$ then the smallest period of Y_n is at least q_t .*

Proof. It suffices to prove the result for $n = q_t + q_{t-1} - 1$, since any period of $Y_{n'}$, $n' > n$, is at least as large as the smallest period of Y_n . Write $Y_{n+1} = X_t X_{t-1}$, where $|X_t| = q_t$ and $|X_{t-1}| = q_{t-1}$. By the ‘‘almost commutative’’ property stated previously, if ab are the last two letters of $X_t X_{t-1}$, we have $Y_{n-1} = X_t X_{t-1}(ab)^{-1} = X_{t-1} X_t (ba)^{-1}$. By Proposition 9, the word Y_{n-1} is a central word and has periods q_t and q_{t-1} , with q_{t-1} being its smallest period. Since X_{t-1} is a prefix of X_t , it is clear that Y_n has period q_t . The word Y_n does not have period q_{t-1} , since it would then be a word of length $q_t + q_{t-1} - 1$ with co-prime periods q_t and q_{t-1} , contrary to the Fine-Wilf theorem. The word Y_n therefore does not have any period that is a multiple of q_{t-1} . Furthermore, if Y_n had a period q with $q_{t-1} < q < q_t$ and q not a multiple of q_{t-1} , then the central word Y_{n-1} would have period q as well. The word Y_{n-1} would then have periods q and q_{t-1} , again violating the Fine-Wilf theorem. It follows that Y_n has smallest period q_t . \square

Lemma 11. *We have $\text{PER}(n) \subseteq A(n)$.*

Proof. The proof is by induction on n . Certainly the result holds for $n = 1$. Suppose the lazy Ostrowski representation of n is $\sum_{0 \leq i \leq t} d_i q_i$. By Lemma 7 we have $q_t + q_{t-1} - 1 \leq n \leq$

$q_{t+1} + q_t - 2$. Suppose that the elements of $A(n)$ are ordered by size and note that q_t and n are the least and greatest elements of $A(n)$ respectively.

By Lemma 10, the minimal period of Y_n is at least q_t , and clearly the maximal period of Y_n is n . Consequently, if there is some $p \in \text{PER}(n)$ such that $p \notin A(n)$, then there are two consecutive periods $p_1, p_2 \in A(n)$ such that $p_1 < p < p_2$. We find then that Y_{n-p_1} has periods $p_2 - p_1$ and $p - p_1$.

By the definition of $A(n)$, the period p_1 has the form

$$p_1 = d_t q_t + d_{t-1} q_{t-1} + \cdots + d_{j+1} q_{j+1} + a q_j$$

for some $a \leq d_j$. Hence $n - p_1$ has lazy representation (possibly including some leading 0's) $(d_j - a)d_{j-1} \cdots d_0$. By the induction hypothesis, we have $\text{PER}(n - p_1) \subseteq A(n - p_1)$. However, since p_2 and p_1 are consecutive periods of Y_n , we have $p_2 - p_1 = q_j$ if $a < d_j$ or $p_2 - p_1 = q_{j'}$, where j' is the largest index $< j$ such that $d_{j'} > 0$, if $a = d_j$. By the definition of $A(n - p_1)$, the least element of $A(n - p_1)$ is q_j if $a < d_j$ or $q_{j'}$ if $a = d_j$. It follows that $p_2 - p_1$ is the least element of $A(n - p_1)$. However, $p - p_1$ is smaller than $p_2 - p_1$, so we have $p - p_1 \in \text{PER}(n - p_1)$ but $p - p_1 \notin A(n - p_1)$ which is a contradiction. \square

Theorem 6 now follows from Lemmas 8 and 11.

Let us now apply these results to the infinite Fibonacci word $\mathbf{f} = 01001010 \cdots$, which equals the Sturmian characteristic word \mathbf{x}_α for $\alpha = (3 - \sqrt{5})/2 = [0, 2, 1, 1, 1, \dots]$. Recall that the n 'th Fibonacci number is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. An easy induction shows that $q_i = F_{i+2}$ for $i \geq 0$. Here the ordinary Ostrowski representation corresponds to the familiar and well-studied Fibonacci (or Zeckendorf) representation [15, 24] as a sum of distinct Fibonacci numbers. The lazy Ostrowski representation, on the other hand, corresponds to the so-called ‘‘lazy Fibonacci representation’’, as studied by Brown [6]. This representation has the property that it contains no two consecutive 0's.

Theorem 6 now has the following implications for the Fibonacci word.

Corollary 12.

- (a) *If the lazy Fibonacci representation of n is $n = F_{t_1} + F_{t_2} + \cdots + F_{t_r}$, for $t_1 < t_2 < \cdots < t_r$, then the periods of the length- n prefix of the Fibonacci word are*

$$F_{t_r}, F_{t_r} + F_{t_{r-1}}, F_{t_r} + F_{t_{r-1}} + F_{t_{r-2}}, \dots, F_{t_r} + F_{t_{r-1}} + \cdots + F_{t_1}.$$

- (b) *The shortest prefix of \mathbf{f} having exactly n periods (including the trivial period) is of length $F_{n+3} - 2$, for $n \geq 1$.*
- (c) *The longest prefix of \mathbf{f} having exactly n periods (including the trivial period) is of length $F_{2n+2} - 1$, for $n \geq 1$.*
- (d) *The least period of $\mathbf{f}[0..m - 1]$ is F_n for $F_{n+1} - 1 \leq m \leq F_{n+2} - 2$ and $n \geq 2$.*

Proof.

- (a) This is just a restatement of Theorem 6 for the special case $\alpha = (3 - \sqrt{5})/2$.
- (b) This corresponds to the lazy Fibonacci representation $\overbrace{11 \cdots 1}^n$, which equals the sum $F_2 + F_3 + \cdots + F_{n+1}$, for which a classical Fibonacci identity gives $F_{n+3} - 2$.
- (c) This corresponds to the lazy Fibonacci representation $(10)^n$, which equals the sum $F_3 + F_5 + \cdots + F_{2n+1}$, for which a classical Fibonacci identity gives $F_{2n+2} - 1$.
- (d) Theorem 6 implies that the least period of every n with Ostrowski representation of length t is F_{t+1} . Lemma 7 implies that $q_{t-1} + q_{t-2} - 1 \leq n \leq q_t + q_{t-1} - 2$; in other words, $F_{t+1} + F_t - 1 \leq n \leq F_{t+2} + F_{t+1} - 2$, or $F_{t+2} - 1 \leq n \leq F_{t+3} - 2$.

□

For another connection between Ostrowski numeration and periods of Sturmian words, see [21]. Saari [20] determined the least period of every factor of the Fibonacci word, not just the prefixes; also see [18, Thm. 3.15].

4 Tightness of the period inequality

Returning to our period inequality, it is natural to wonder if the bound (1) is tight. We exhibit a class of binary words for which it is.

Let g_s , for $s \geq 1$, be the prefix of length $F_{s+2} - 2$ of \mathbf{f} . Thus, for example, $g_1 = \epsilon$, $g_2 = 0$, $g_3 = 010$, $g_4 = 010010$, and so forth. We now show that the bound (1) is tight, up to an additive factor, for the words g_s . Let $\tau = (1 + \sqrt{5})/2$, the golden ratio.

Theorem 13. *Take $x = g_s$ for $s \geq 4$. Then the left-hand side of (1) is $s - 2$, while the right-hand side is asymptotically $s + c$ for $c = 3 + \tau^2/2 - (\ln 2\sqrt{5})/(\ln \tau) \doteq 1.19632$.*

Proof. Take $x = g_s$. By definition we have $n = |x| = F_{s+2} - 2$. By Corollary 12 (b) we know that g_s has $s - 1$ periods, and hence $s - 2$ nontrivial periods. Thus $\text{nnp}(x) = s - 2$.

Next let's compute $\text{ice}(g_s)$. Corollary 12 (d) states that the least period of the prefix $\mathbf{f}[0..m - 1]$ equals F_s for $F_{s+1} - 1 \leq m \leq F_{s+2} - 2$, $s \geq 2$. It follows that the exponent of the prefix $\mathbf{f}[0..m - 1]$ is m/F_s for $F_{s+1} - 1 \leq m \leq F_{s+2} - 2$, $s \geq 2$. For fixed s , the quantity m/F_s is maximized at $m = F_{s+2} - 2$, which gives an exponent of $(F_{s+2} - 2)/F_s$. It remains to see that the sequence $((F_{s+2} - 2)/F_s)_{s \geq 2}$ is strictly increasing. For this it suffices to show that $(F_{s+2} - 2)/F_s < (F_{s+3} - 2)/F_{s+1}$ for $s \geq 2$, or, equivalently,

$$F_{s+2}F_{s+1} - F_sF_{s+3} < 2F_{s+1} - 2F_s. \quad (6)$$

But an easy induction shows that the left-hand side of (6) is $(-1)^s$, while the right-hand side is $2F_{s-1} \geq 2$. Thus we see $\beta = \text{ice}(g_s) = (F_{s+2} - 2)/F_s$.

Hence the right-hand side of (1) is

$$\frac{F_{s+2} - 2}{2F_s} + 1 + \frac{\ln((F_{s+2} - 2)/2)}{\ln(\frac{F_{s+2}-2}{F_{s+1}-2})}.$$

Now use the Binet formula for Fibonacci numbers, which implies that $F_s \sim \tau^s/\sqrt{5}$, and the fact that $\lim_{s \rightarrow \infty} F_s/F_{s-1} = \tau$, to obtain that the right-hand side of (1) is asymptotically

$$\frac{\tau^2}{2} + 1 + (s + 2) - (\ln 2\sqrt{5})/(\ln \tau).$$

This gives the desired result. □

5 Two measures of periodicity

Corollary 2 (b) suggests that the quantity

$$M(x) := \frac{\text{np}(x)}{\text{ice}(x) \ln |x|}$$

is a measure of periodicity for finite words x . It also suggests studying the following measures of periodicity for infinite words \mathbf{x} . For $n \geq 2$ let Y_n be the prefix of length n of \mathbf{x} . Then define

$$P(\mathbf{x}) := \limsup_{n \rightarrow \infty} M(Y_n)$$

$$p(\mathbf{x}) := \liminf_{n \rightarrow \infty} M(Y_n)$$

From Theorem 4, we know that for the “typical” infinite word \mathbf{x} we have $P(\mathbf{x}) = p(\mathbf{x}) = 0$. Thus it is of interest to find words \mathbf{x} where $P(\mathbf{x})$ and $p(\mathbf{x})$ are large. In this section we compute these measures for several infinite words.

Theorem 14. *Let \mathbf{f} denote the Fibonacci infinite word. Then $P(\mathbf{f}) = 1/(\tau^2 \ln \tau) \doteq 0.79375857$ and $p(\mathbf{f}) = 1/(2\tau^2 \ln \tau) \doteq 0.396879286$.*

Proof. This follows immediately from Corollary 12, together with the calculation of ice given in the proof of Theorem 13. □

The *period-doubling word* \mathbf{d} is defined to be the fixed point of the morphism sending $1 \rightarrow 10$ and $0 \rightarrow 11$; see [10].

Theorem 15. $P(\mathbf{d}) = \frac{1}{2 \ln 2} \doteq 0.7213$ and $p(\mathbf{d}) = \frac{1}{4 \ln 2} \doteq 0.36067$.

Proof. Since \mathbf{d} is not a Sturmian word, or even closely related to one, we need to use different techniques from those we used previously.

Let $r(n)$ denote the number of periods (including the trivial period) in the length- n prefix of \mathbf{d} . We use $(n)_2$ to denote the canonical base-2 representation of n , and $(n, p)_2$ to denote the base-2 representation of n and p as a sequence of pairs of bits (where the shorter representation is padded with leading zeros, if necessary).

We can use the theorem-proving software `Walnut` to calculate the periods of prefixes of \mathbf{d} . (For more about `Walnut`, see [17].) We sketch the ideas briefly.

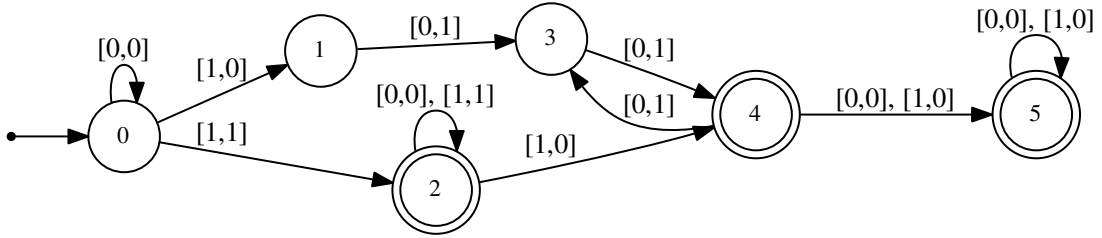
We can write a first-order logical formula $\text{pdp}(m, p)$ stating that the prefix of length $m \geq 1$ of \mathbf{d} has period p , $1 \leq p \leq m$:

$$\begin{aligned} \text{pdp}(m, p) &:= (1 \leq p \leq m) \wedge \mathbf{d}[0..m-p-1] = \mathbf{d}[p..m-1] \\ &= (1 \leq p \leq m) \wedge \forall t (0 \leq t < m-p) \implies \mathbf{d}[t] = \mathbf{d}[t+p]. \end{aligned}$$

Such a formula can be automatically translated, using `Walnut`, to an automaton that recognizes the language

$$\{(n, p)_2 : \text{the length-}n \text{ prefix of } \mathbf{d} \text{ has period } p\}.$$

We depict it below.



Such an automaton can be automatically converted by `Walnut` to a linear representation for $r(n)$, as discussed in [8]. This is a triple (v, ρ, w) where v, w are vectors, and ρ is a matrix-valued morphism, such that $r(n) = v \cdot \rho((n)_2) \cdot w$. The values are given below:

$$v = [100000] \quad \rho(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho(1) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

From this, using the technique described in [14], we can easily compute the relations

$$\begin{aligned} r(0) &= 0 \\ r(2n + 1) &= r(n) + 1, \quad n \geq 0 \\ r(4n) &= r(n) + 1, \quad n \geq 1 \\ r(4n + 2) &= r(n) + 1, \quad n \geq 0. \end{aligned}$$

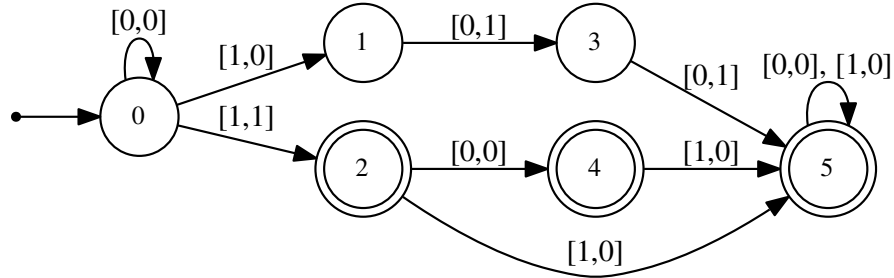
Reinterpreting this definition for r , we see that $r(n)$ is equal to the length of the (unique) factorization of $(n)_2$ into the factors 1, 00, and 10. It now follows that

- (a) The smallest m such that $r(m) = n$ is $m = 2^n - 1$;
- (b) The largest m such that $r(m) = n$ is $m = \lfloor 2^{2n+1}/3 \rfloor$, with $(m)_2 = (10)^n$.

Similarly, we can use `Walnut` to determine the smallest period p of every length- n prefix of \mathbf{d} . We use the predicate

$$\text{pdlp}(n, p) := \text{pdp}(n, p) \wedge \forall q (1 \leq q < p) \implies \text{pdp}(n, q).$$

This gives the automaton



Inspection of this automaton shows that least period of the prefix of length n is, for $s \geq 2$, equal to $3 \cdot 2^{s-2}$ for $2^s \leq n < 5 \cdot 2^{s-2}$ and 2^s for $5 \cdot 2^{s-2} \leq n < 2^{s+1}$. It follows that the initial critical exponent of every prefix of \mathbf{d} of length n , for $2^t - 1 \leq n \leq 2^{t+1} - 2$, is $2 - 2^{1-t}$.

The result now follows. □

Let $\mathbf{t} = t_0 t_1 t_2 \dots = 01101001 \dots$ be the Thue-Morse word, the fixed point of the morphism μ defined by $\mu(0) = 01$ and $\mu(1) = 10$.

Theorem 16. *We have $P(\mathbf{t}) = 3/(10 \ln 2) \doteq 0.4328$ and $p(\mathbf{t}) = 0$.*

Proof. We have $\text{ice}(x) = 5/3$ for every prefix x of \mathbf{t} of length ≥ 5 , a claim that can easily be verified with `Walnut`.

For the value of $p(\mathbf{t})$, it suffices to observe that $\text{np}(x) = 1$ if x is a prefix of \mathbf{t} of length $3 \cdot 2^n + 1$ for $n \geq 0$, which can also be verified with `Walnut`.

For $P(\mathbf{t})$ it suffices to show that the shortest prefix of \mathbf{t} having n nontrivial periods is of length $2^{2n-1} + 2$. For this we can use `Walnut`, but the analysis is somewhat complicated. Letting $v(n)$ denote the number of nontrivial periods of the length- n prefix of \mathbf{t} , we can mimic what we did for the period-doubling word, obtaining the matrices and the following relations for $n \geq 0$:

$$\begin{aligned} v(4n) &= v(n) + [n \neq 0] \\ v(4n + 3) &= v(4n + 1) \\ v(8n + 1) &= v(2n + 1) + t_n \\ v(8n + 2) &= v(2n + 1) + t_n \\ v(8n + 6) &= v(4n + 1) + 1 - t_n \\ v(16n + 5) &= v(2n + 1) + 1 \\ v(16n + 13) &= v(4n + 1) + 1. \end{aligned}$$

Here $[n \neq 0]$ is the Iverson bracket, which evaluates to 1 if the condition holds and 0 otherwise.

Now a tedious induction on m , which we omit, shows that

$$\begin{aligned} m \text{ is even and } v(m) \geq n &\implies m \geq 2^{2n-3} + 2; \\ m \text{ is odd and } v(m) \geq n &\implies m \geq 2^{2n-2} + 1, \end{aligned}$$

and furthermore $v(2^{2n-3} + 2) = n$ for $n \geq 2$. It follows that the shortest prefix of \mathbf{t} having n nontrivial periods is of length $2^{2n-1} + 2$ for $n \geq 2$, from which the desired result follows. \square

Remark 17. The `Walnut` commands for the last two results are available on the third author's web page, at

<https://cs.uwaterloo.ca/~shallit/papers.html> .

`Walnut` itself is available at

<https://github.com/hamousavi/Walnut> .

Remark 18. It would be interesting to compute the values of

$$\begin{aligned} D_1 &:= \inf_{n \geq 1} \sup_{x \in \{0,1\}^n} M(x) \\ D_2 &:= \liminf_{n \rightarrow \infty} \sup_{x \in \{0,1\}^n} M(x). \end{aligned}$$

Theorem 14 shows that $D_2 \geq 1/(2\tau^2 \ln \tau) \doteq 0.396879286$. Thus, for example, for every sufficiently large n there is a length- n binary string x with $M(x) \geq .396$.

6 Shortest overlap-free binary word with p periods

In this section and the following one, we consider how quickly the number of periods can grow if we enforce an upper bound on the exponent of repetitions occurring in the word.

Recall that an *overlap* is a word of the form $axaxa$, where a is a single letter and x is a (possibly empty) word. An example in English is the word **alfalfa**. We say a word is *overlap-free* if no finite factor is an overlap.

Define $f(p)$ to be the length of the shortest binary overlap-free word having p nontrivial periods. Recall that we call a border w of x *short* if $|w| < |x|/2$.

If $w = axa$ for a single letter a and (possibly empty) word x , define $\gamma(w) = a^{-1}\mu^2(w)a^{-1}$, or, in other words, the word $\mu^2(w)$ with an a removed from the front and back.

Lemma 19. *Define a sequence of words $(A_n)_{n \geq 3}$ as follows:*

$$A_n = \begin{cases} 001001100100, & \text{if } n = 3; \\ \gamma(A_{n-1}), & \text{if } n \geq 4. \end{cases}$$

Then A_n is a palindrome with n short palindromic borders for $n \geq 3$.

Proof. Observe that if w is a palindrome, then so is $\gamma(w)$. Write $\bar{a} = 1 - a$ for $a \in \{0, 1\}$.

We now prove the claim by induction on n . It is true for $n = 3$, since the borders are $0, 00$, and 00100 .

Now assume the result is true for n ; we prove it for $n + 1$. Suppose n short palindromic borders of A_n are w_1, w_2, \dots, w_n , and each starts with the letter a . From the observation above, we know that $A_{n+1} = \gamma(A_n)$ is a palindrome. We claim that $\bar{a}, \gamma(w_1), \gamma(w_2), \dots, \gamma(w_n)$ are short palindromic borders of $\gamma(A_n)$.

To see that \bar{a} is a border of A_{n+1} , note that $A_n = awa$ for some w , so $\gamma(A_n) = \bar{a}\bar{a}a\mu^2(w)a\bar{a}\bar{a}$.

Otherwise, let w_i be a palindromic border of A_n . Since it is short, we have $A_n = w_i y w_i$ for some y . Then $\gamma(w_i)$ is both a prefix and suffix of $\gamma(A_n)$ and hence is a palindromic border of A_{n+1} . The shortness of the borders is trivial.

Thus A_{n+1} has at least $n + 1$ palindromic short borders. □

Corollary 20. *We have $f(1) = 2$, $f(2) = 5$, and $f(p) \leq (17/6)4^{p-2} + 2/3$ for $p \geq 3$.*

Proof. For $p = 1$, the shortest binary overlap-free word with 1 nontrivial period is 00 . For $p = 2$ it is 00100 .

Next we argue, by induction on p , that each A_p , for $p \geq 3$, is overlap-free. The base case is $p = 3$, and is easy to check. Otherwise assume the result is true for A_p . We now use a classical result that if a word x is overlap-free, then so is $\mu(x)$ [23]. Applying this twice, we see that $\mu^2(A_p)$ is overlap-free. Then $A_{p+1} = \gamma(A_p)$ is overlap-free, since it is a factor of $\mu^2(A_p)$.

As we have seen above, A_p has p borders and hence p nontrivial periods. The only thing left to verify is that $|A_p| = (17/6)4^{p-2} + 2/3$ for $p \geq 3$. This is an easy induction, and is left to the reader. □

Remark 21. One can go from A_p to A_{p+1} , for $p \geq 3$, via the following procedure, which we state without proof. Write A_p in terms of its run-length encoding, that is, $A_p = a^{e_1}b^{e_2}a^{e_3}b^{e_4} \dots$, where $a \neq b$ and all the e_i are positive. Then, considering c^e as the pair (c, e) , apply the following morphism:

$$\begin{aligned} (0, 1) &\rightarrow 1101 \\ (1, 1) &\rightarrow 0010 \\ (0, 2) &\rightarrow 11001101 \\ (1, 2) &\rightarrow 00110010 \end{aligned}$$

Finally, drop the last two symbols.

Remark 22. We conjecture that the words A_p constructed above are actually the shortest overlap-free binary words with p periods with $p \geq 3$, but we do not currently have a proof of this claim in general. The sequence $(f(p))$ is sequence [A334811](#) in the *On-Line Encyclopedia of Integer Sequences* [22].

7 Shortest squarefree ternary word with p periods

Recall that a *square* is a nonempty word of the form xx , such as the English word *murmur*. A word is *squarefree* if no finite factor is a square.

Let $g(p)$ be the length of the shortest ternary squarefree word having p nontrivial periods. Here are the first few values of g , computed through exhaustive search.

p	0	1	2	3	4
$g(p)$	1	3	7	23	59

Theorem 23. For $p \geq 3$ we have $g(p) \leq \frac{17}{12}4^{p-1} + 1/3$.

Proof. Consider the words A_p defined above. Suppose A_p starts and ends with the letter a . Let B_p be the word whose i 'th letter is the number of occurrences of \bar{a} between the i 'th and the $(i + 1)$ 'th occurrence of a . For example, we have

$$B_3 = 0102010$$

$$B_4 = 02012102012021020121020$$

$$B_5 = 020120210201210120210121020120210201210120210201202101210201202102012101202101210201202102012021020$$

Then each B_p is squarefree. For if B_p had a square, say $c_1c_2 \dots c_t c_1c_2 \dots c_t$, then A_p has the overlap

$$ab^{c_1}ab^{c_2} \dots ab^{c_t}ab^{c_1}ab^{c_2} \dots ab^{c_t}a,$$

where $b = \bar{a}$, a contradiction.

Furthermore, each border of A_p , except the border of length 1, corresponds via this map to a border of B_p . So $\text{npb}(B_p) = p - 1$. By induction we can show $|A_p| = |B_p|/2 = (17/12)4^{p-2} + 1/3$ for $p \geq 4$. It follows that $g(p) \leq (17/12)4^{p-1} + 1/3$. \square

Remark 24. Our bound is clearly not optimal. It would be interesting to obtain better bounds for $g(p)$. The sequence $(g(p))$ is sequence [A332866](#) in the *On-Line Encyclopedia of Integer Sequences* [22].

Remark 25. One can go from B_p to B_{p+1} , for $p \geq 4$, using the following procedure, which we state without proof. Take B_p and replace every other 1 in it with 3. Then apply the following morphism:

$$\begin{aligned} 0 &\rightarrow 0201 \\ 1 &\rightarrow 2101 \\ 2 &\rightarrow 2021 \\ 3 &\rightarrow 0121. \end{aligned}$$

Finally, drop the last letter.

Acknowledgments

Thanks to Anna Frid, Jean-Paul Allouche, Luke Schaeffer, Kalle Saari, Štěpán Holub, Jean Berstel, and Valérie Berthé for their helpful comments.

We also thank the referees, one of whom provided a substantial simplification of the proof of the inclusion $A(n) \subseteq \text{PER}(n)$.

References

- [1] J.-P. Allouche and J. O. Shallit, *Automatic Sequences*, Cambridge University Press, 2003.
- [2] J. Berstel. Sur la construction de mots sans carré. *Séminaire de Théorie des Nombres* (1978–1979), 18.01–18.15.
- [3] J. Berstel, A. Lauve, C. Reutenauer, and F. V. Saliola. *Combinatorics on Words*. CRM Monograph Series, Vol. 27, American Mathematical Society, 2009.
- [4] J. Berstel and P. Séébold. Sturmian words. In M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, 2002, pp. 45-110.
- [5] V. Berthé, C. Holton, and L. Q. Zamboni. Initial powers of Sturmian sequences. *Acta Arithmetica* **123** (2006), 315–347.
- [6] J. L. Brown, Jr. A new characterization of the Fibonacci numbers. *Fib. Quart.* **3** (1965) 1–8.

- [7] A. Carpi and A. de Luca. Central Sturmian words: recent developments. In C. De Felice and A. Restivo, eds., *DLT 2005*, Lect. Notes in Computer Sci., Vol. 3572, Springer-Verlag, 2005, pp. 36–56.
- [8] E. Charlier, N. Rampersad, and J. Shallit. Enumeration and decidable properties of automatic sequences. *Internat. J. Found. Comp. Sci.* **23** (2012), 1035–1066.
- [9] J. D. Currie and K. Saari. Least periods of factors of infinite words. *RAIRO Inform. Théor. App.* **43** (2009), 165–178.
- [10] D. Damanik. Local symmetries in the period doubling sequence. *Discrete Appl. Math.* **100** (2000) 115–121.
- [11] C. Epifanio, C. Frougny, A. Gabriele, F. Mignosi, and J. Shallit. Sturmian graphs and integer representations over numeration systems. *Disc. Appl. Math.* **160** (2012), 536–547.
- [12] N. J. Fine and H. S. Wilf. Uniqueness theorems for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965), 109–114.
- [13] A. E. Frid. Sturmian numeration systems and decompositions to palindromes. *European J. Combin.* **71** (2018) 202–212.
- [14] D. Goč, H. Mousavi, and J. Shallit. On the number of unbordered factors. In A.-H. Dediu, C. Martin-Vide, and B. Truthe, editors, *LATA 2013*, Vol. 7810 of *Lecture Notes in Computer Science*, pp. 299–310. Springer-Verlag, 2013.
- [15] C. G. Lekkerkerker. Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. *Simon Stevin* **29** (1952), 190–195.
- [16] R. C. Lyndon and M. P. Schützenberger. The equation $a^M = b^N c^P$ in a free group. *Michigan Math. J.* **9** (1962), 289–298.
- [17] H. Mousavi. Automatic theorem proving in Walnut. Available at <http://arxiv.org/abs/1603.06017>, 2016.
- [18] H. Mousavi, L. Schaeffer, and J. Shallit. Decision algorithms for Fibonacci-automatic words, I: basic results. *RAIRO Inform. Théorique* **50** (2016), 39–66.
- [19] C. Reutenauer. *From Christoffel Words to Markoff Numbers*. Oxford University Press, 2019.
- [20] K. Saari. Periods of factors of the Fibonacci word. In *Proc. 6th International Conference on Words (WORDS '07)*, Institut de Mathématiques de Luminy, 2007, pp. 273–279.

- [21] L. Schaeffer. Ostrowski numeration and the local period of Sturmian words. In A.-H. Dediu, C. Martín-Vide, and B. Truthe, eds., *LATA 2013*, Lect. Notes in Comp. Sci., Vol. 7810, Springer, 2013, pp. 493–503.
- [22] N. J. A. Sloane et al. The On-Line Encyclopedia of Integer Sequences. Available at <https://oeis.org>, 2020.
- [23] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912), 1–67.
- [24] E. Zeckendorf. Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. *Bull. Soc. Roy. Liège* **41** (1972), 179–182.