UNIVERSITY of GUELPH

CIS1910 Discrete Structures in Computing (I)
Winter 2019, Solutions to Assignment 3

A. Rules of Inference

1) Let d be the proposition "I'm dreaming", let h be the proposition "I'm hallucinating", and let e be the proposition "I see elephants running down the road". We want to show that the premises $d \vee h$, $\neg d$ and $h \rightarrow e$ imply e.

Step	**Reason**
(1) $d \vee h$	Premise
(2) $\neg d$	Premise
(3) h	Disjunctive syllogism using (1) and (2)
(4) $h \rightarrow e$	Premise
(5) e	Modus ponens using (3) and (4)

2) Let A be the proposition "Allen is a good boy", let H be the proposition "Hillary is a good girl", and let D be the proposition "David is happy". We want to show that the premises $\neg A \vee H$ and $A \vee D$ imply $H \vee D$.

Step	**Reason**
(1) $\neg A \vee H$	Premise
(2) $A \vee D$	Premise
(3) $D \vee H$	Resolution using (2) and (1)
(4) $H \vee D$	Using (3) and the fact that $\vee$ is commutative

3)

Step	**Reason**
(1) $q\to(u\wedge t)$	Premise
(2) q	Premise
(3) $u\wedge t$	Modus ponens using (2) and (1)
(4) u	Simplification using (3)
(5) $t\wedge u$	Using (3) and the fact that $\wedge$ is commutative
(6) t	Simplification using (5)
(7) $u\to p$	Premise
(8) p	Modus ponens using (4) and (7)
(9) $p\wedge t$	Conjunction using (8) and (6)
(10) $(p\wedge t)\to(r\vee s)$	Premise
(11) $r\vee s$	Modus ponens using (9) and (10)
(12) $s\vee r$	Using (11) and the fact that $\vee$ is commutative
(13) $\neg s$	Premise
(14) r	Disjunctive syllogism using (12) and (13)

4) Let C be the unary predicate whose domain is the set P of all people and such that C(p) is the statement “p is in this class”, let F be the unary predicate whose domain is P and such that F(p) is the statement “p has been to France”, and let L be the unary predicate whose domain is P and such that L(p) is the statement “p has visited the Louvre”. We want to show that the premises $\exists p, (C(p)\wedge F(p))$ and $\forall p, (F(p)\to L(p))$ imply $\exists p, (C(p)\wedge L(p))$. In the proof below, q represents an unspecified particular person.

Step	**Reason**
(1) $\exists p, (C(p)\wedge F(p))$	Premise
(2) $C(q)\wedge F(q)$	Existential instantiation using (1)
(3) $C(q)$	Simplification using (2)
(4) $F(q)$	Simplification using (2) and the fact that $\wedge$ is commutative
(5) $\forall p, (F(p)\to L(p))$	Premise
(6) $F(q)\to L(q)$	Universal instantiation using (5)
(7) $L(q)$	Modus ponens using (4) and (6)
(8) $C(q)\wedge L(q)$	Conjunction using (3) and (7)
(9) $\exists p, (C(p)\wedge L(p))$	Existential generalization using (8)

B. Direct Proofs

5) Let m and n be two odd integers. There exist two integers p and q such that $m=2p+1$ and $n=2q+1$. Therefore, $mn=(2p+1)(2q+1)=4pq+2p+2q+1=2(2pq+p+q)+1$. We have found an integer r (the integer $r=2pq+p+q$) such that $mn=2r+1$. In other words, mn is odd.

6) Consider three integers a, b and c, with $a\neq0$ and $b\neq0$. Assume $a|b$ and $b|c$. Then, there exist two integers m and n such that $b=am$ and $c=bn$. Therefore, $c=(am)n=a(mn)$. We have found an integer p ($p=mn$) such that $c=ap$. In other words, $a|c$.

C. Proofs by Contraposition

7) Consider two real numbers x and y. Assume x and y are rational numbers. Then, there exist four integers p, q, r and s such that x=p/q and y=r/s. Therefore, xy=(p/q)(r/s)=(pr)/(qs). We have found two integers u and v (u=pr and v=qs) such that xy=u/v. In other words, xy is a rational number. By contraposition, if xy is an irrational number then x or y is an irrational number.

8) Consider an integer n. Assume n is even. Then, there exists an integer p such that n=2p. Therefore, $n^5+7=(2p)^5+7=2\times2^4\times p^5+2\times3+1=2(16p^5+3)+1$. We have found an integer q ($q=16p^5+3$) such that $n^5+7=2q+1$. In other words, n^5+7 is odd. By contraposition, if n^5+7 is even then n is odd.

D. Proofs by Contradiction

9) Let a, b and c be three integers with $a\neq0$. We know that $\neg(p\rightarrow q)\equiv p\wedge\neg q$. In particular, the negation of "if a does not divide bc then a does not divide b" is "a does not divide bc and a divides b". Assume that a does not divide bc and a divides b. Then, there exists an integer m such that $b=am$. Therefore, $bc=amc$. We have found an integer n ($n=mc$) such that $bc=an$. In other words, a divides bc. Since our assumption leads to a contradiction, it must be false.

10) Assume a and b are two nonnegative real numbers such that $(a+b)/2<\sqrt{(ab)}$, i.e., $a+b<2\sqrt{(ab)}$. Since both sides of the inequality are nonnegative real numbers, we have $[a+b]^2<[2\sqrt{(ab)}]^2$, i.e., $a^2+2ab+b^2<4ab$, hence $a^2-2ab+b^2<0$, i.e., $(a-b)^2<0$. However, we know that $(a-b)^2\geq0$. Since our assumption leads to a contradiction, it must be false.

E. Proofs by Induction

For a proof by induction, you should follow the structure given in class.
In the proofs below, pay attention to the sentences in green.

11) $1\times1!=1$, $1\times1!+2\times2!=5$, $1\times1!+2\times2!+3\times3!=23$. It seems that, in general, $1\times1!+2\times2!+\ldots+n\times n!=(n+1)!-1$. This, of course, is only a conjecture. Let us prove it.

PREDICATE
Let P be the unary predicate whose domain is 1..+∞ and such that P(n) is the statement:
"$1\times1!+2\times2!+\ldots+n\times n!=(n+1)!-1$"

BASIS STEP
P(1) is true, since $1\times1!=1$ and $(1+1)!-1=1$.

INDUCTIVE STEP
Let n be an arbitrary element of the domain. Assume P(n) is true. Let us show that P(n+1) is true.

$$\begin{aligned}&1\times1!+2\times2!+\ldots+n\times n!+(n+1)\times(n+1)!\\&=(n+1)!-1+(n+1)\times(n+1)!\\&=(n+2)\times(n+1)!-1\\&=(n+2)!-1\end{aligned}$$

(Note that the first equality derives from the inductive hypothesis, i.e., the assumption that P(n) is true.) We have shown that P(n+1) is true.

CONCLUSION
By induction, we can conclude that: ∀n, P(n)

12)

PREDICATE
Let P be the unary predicate whose domain is 0..+∞ and such that P(n) is the statement:
"$(2n+1)^2-1$ is divisible by 8"

BASIS STEP
P(0) is true, since $(2\times0+1)^2-1=0=8\times0$.

INDUCTIVE STEP
Let n be an arbitrary element of the domain. Assume P(n) is true, i.e., assume there exists an integer p such that $(2n+1)^2-1=8p$. Let us show that P(n+1) is true.

$$\begin{aligned}&(2(n+1)+1)^2-1\\&=((2n+1)+2)^2-1\\&=(2n+1)^2+4(2n+1)+4-1\\&=(2n+1)^2-1+4(2n+2)\\&=8p+8(n+1)\\&=8(p+n+1)\end{aligned}$$

We have found an integer q (the integer $q=p+n+1$) such that $(2(n+1)+1)^2-1=8q$.
In other words, $(2(n+1)+1)^2-1$ is divisible by 8; we have shown that P(n+1) is true.

CONCLUSION
By induction, we can conclude that: ∀n, P(n)

F. Proofs by Strong Induction

The structure of a proof by strong induction is exactly the same as that of a proof by induction. The difference lies in the way the predicate is defined. In the following proofs, pay attention to the sentences in blue.

13) If n=1, no break is needed; if n=2, we need 1 break; if n=3, we need 2 breaks. It seems that, in general, n–1 breaks are needed. This, of course, is only a conjecture. Let us prove it.

PREDICATE
Let P be the unary predicate whose domain is 1..+∞ and such that P(n) represents the statement: "for any k in the domain and less than or equal to n, the number of breaks needed to break a bar of k squares into separate squares is k–1"

BASIS STEP
P(1) is true, because, as we said, no break (i.e., 1–1=0 break) is then needed.

INDUCTIVE STEP
Let n be an arbitrary element of the domain. Assume P(n) is true. Let us show that P(n+1) is true.

According to the inductive hypothesis, for any k in the domain and less than or equal to n, the number of breaks needed to break a bar of k squares into separate squares is k –1.
If we can show that this also holds when k=n+1, then our proof will be complete.

Let us make 1 first break. The bar of n+1 squares is broken into two pieces: one of, say, i squares (where i belongs to 1..n), and one of n+1–i squares (note that n+1–i also belongs to 1..n). According to the inductive hypothesis, we need i–1 breaks to break the first piece into separate squares and n+1–i–1=n–i breaks to break the second piece. In total, we therefore need 1+(i–1)+(n–i)=n breaks to break the bar of n+1 squares into separate squares.
We have shown that P(n+1) is true.

CONCLUSION
By induction, we can conclude that: ∀n, P(n)

14)

PREDICATE
Let P be the unary predicate whose domain is 1..+∞ and such that P(n) is the statement: "for any integer k of the domain less than or equal to n we have $T_k<2^k$"

BASIS STEP
P(1) is true, since $T_1=1<2^1$.

INDUCTIVE STEP
Let n be an arbitrary element of the domain. Assume P(n) is true. Let us show that P(n+1) is true.

According to the inductive hypothesis, for any integer k of the domain less than or equal to n we have $T_k<2^k$. If we can show that this also holds when k=n+1, our proof will be complete.

Since $T_2=T_3=1$, we obviously have $T_{n+1}<2^{n+1}$ when n=1 or n=2. Assume n≥3.
Then, $T_{n+1}=T_n+T_{n-1}+T_{n-2}<2^n+2^{n-1}+2^{n-2}$ according to the inductive hypothesis.
Since $2^n+2^{n-1}+2^{n-2}=2^{n+1}(1/2+1/4+1/8)=2^{n+1}\times 7/8<2^{n+1}$, we have $T_{n+1}<2^{n+1}$.
We have shown that P(n+1) is true.

CONCLUSION
By induction, we can conclude that: ∀n, P(n)

G. Other

15) Consider two real numbers x and y. There are 4 cases.
Case 1: x≥0 and y≥0. Then x+y≥0 and |x+y|=x+y=|x|+|y|
Case 2: x<0 and y<0. Then x+y<0 and |x+y|=–(x+y)=(–x)+(–y)=|x|+|y|
Case 3: x≥0 and y<0. There are 2 subcases.
Case 3a: x≥–y. Then x+y≥0 and |x+y|=x+y<x+(–y)=|x|+|y|
Case 3b: x<–y. Then x+y<0 and |x+y|=–(x+y)=(–x)+(–y)≤x+(–y)=|x|+|y|
Case 4: x<0 and y≥0. Identical to *Case 3* with the roles of x and y reversed.
We have shown that in each possible case we have |x+y|≤|x|+|y|.

16) Let x and y be two real numbers such that x<y. Then x+x<x+y and (x+x)/2<(x+y)/2, i.e., x<(x+y)/2. Moreover, x+y<y+y and (x+y)/2<(y+y)/2=y. In the end, we have found a real number z (the number z=(x+y)/2) such that x<z<y.

17) The cubes of the positive integers less than 10 are 1, 8, 27, 64, 125, 216, 343, 512 and 729. The program below exhausts all the possibilities, and its output is "none of the 120 cases".

```
#include <stdio.h>
int main (void) {
    int cube[]={0,1,8,27,64,125,216,343,512,729};
    int i, j, k, l=0;
    for(i=1;i<=9;i++)
        for(j=1;j<i;j++)
            for(k=j;k<i;k++) {
                l++;
                if(cube[i]==cube[j]+cube[k]) {
                    printf("%d^3 = %d^3 + %d^3\n",i,j,k);
                    return 0;
                }
            }
    printf("none of the %d cases\n",l);
    return 0;
}
```