UNIVERSITY of GUELPH

CIS1910 Discrete Structures in Computing (I)
Winter 2019, Assignment 4

All answers must be justified in a clear, concise and complete manner. If two answers require the same explanations, justify your first answer only, and refer the reader to that justification for the second answer.

DEFINITIONS

(a) Let $\mathfrak{R}$ be the triple (U,V,G), where U and V are two sets and G is a subset of U×V.
We then say that $\mathfrak{R}$ is a ***binary relation over*** U ***and*** V.

For any element u of U and for any element v of V,
if (u,v) belongs to G we say that u is related to v by $\mathfrak{R}$ and we write **$u\mathfrak{R}v$**.

The ***inverse*** of $\mathfrak{R}$ is the relation $\mathfrak{R}^{-1}$=(V,U,G^{-1}) where $G^{-1}=\{(v,u)\in V\times U \mid (u,v)\in G\}$.

Note that a function from U to V (see slide 1.16) is a binary relation over U and V.
The inverse of a function f is the inverse f^{-1} of the relation f.

(b) Let U be a set and let $\mathfrak{R}$ be a binary relation over U and U.
We then say that $\mathfrak{R}$ is a ***binary relation on*** U.

$\mathfrak{R}$ is ***reflexive*** iff: $\forall u\in U,\ u\mathfrak{R}u$
$\mathfrak{R}$ is ***symmetric*** iff: $\forall (u,v)\in U^2,\ (u\mathfrak{R}v \rightarrow v\mathfrak{R}u)$
$\mathfrak{R}$ is ***antisymmetric*** iff: $\forall (u,v)\in U^2,\ [(u\mathfrak{R}v \wedge v\mathfrak{R}u) \rightarrow u=v]$
$\mathfrak{R}$ is ***transitive*** iff: $\forall (u,v,w)\in U^3,\ [(u\mathfrak{R}v \wedge v\mathfrak{R}w) \rightarrow u\mathfrak{R}w]$

(c) Let $\mathfrak{R}$ be a binary relation on a set U. Assume $\mathfrak{R}$ is reflexive, symmetric and transitive.
We then say that $\mathfrak{R}$ is an ***equivalence relation*** on U.
The ***equivalence class*** of an element u of U is the set $\{v\in U \mid u\mathfrak{R}v\}$;
we also say about this set that it is an equivalence class of $\mathfrak{R}$.

(d) Let $\mathfrak{R}$ be a binary relation on a set U. Assume $\mathfrak{R}$ is reflexive, antisymmetric and transitive.
We then say that $\mathfrak{R}$ is an ***order relation*** on U.
The order relation is ***total*** iff: $\forall (u,v)\in U^2,\ (u\mathfrak{R}v \vee v\mathfrak{R}u)$

(e) A function is ***total*** iff its domain and domain of definition are equal.
A function is ***surjective*** iff its range and codomain are equal.
A function is ***injective*** iff its inverse is a function.
A function is ***bijective*** iff it is total, surjective and injective; a ***bijection*** is a bijective function.

PART A. (2+5+3 marks)

1. Let f=(A,B,F) be a total function.
(a) Show that if f is injective then $\forall(x_1,x_2)\in A^2, (f(x_1)=f(x_2) \rightarrow x_1=x_2)$.
(b) Show that if $\forall(x_1,x_2)\in A^2, (f(x_1)=f(x_2) \rightarrow x_1=x_2)$ then f is injective.

2. Consider a bijection f=(A,B,F).
Show that f^{-1} is a bijection from B to A and that for any element x of A we have: $f^{-1}(f(x))=x$.

3. Consider a bijection f from A to B and a bijection g from B to C.
Show that the function $h : A \rightarrow C$, $x \mapsto g(f(x))$ is a bijection. (*Hint:* Use **A1a** and **A1b**.)

PART B. (5×5 marks)

4. Consider the following functions, where I and J denote two subsets of the set $\mathbb{R}$ of real numbers.

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto 1/x \qquad\qquad f_{(I,J)} : I \rightarrow J, \quad x \mapsto f(x)$$

(a) What is the domain of definition of f?

(b) Let y be an element of the codomain of f. Solve the equation f(x)=y in x.
Note that you may have to consider different cases, depending on y.

(c) What is the range of f?

(d) Is f total, surjective, injective, bijective?

(e) Find a pair (I,J) such that $f_{(I,J)}$ is bijective and its range is the range of f.
What is then the inverse of $f_{(I,J)}$?

5. Same questions as above, but with the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x^2$

6. Same questions as above, but with the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \sqrt{x}$

7. Same questions as above, but with the function $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto |x|$

8. Same questions as above, but with the function $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto 1/\sqrt{(x+1)}$

PART C. (4×5 marks)

9. Let $\Re$ be binary relation on $\mathbb{R}$ defined as follows: $\forall(x,y)\in\mathbb{R}^2$, $(x\Re y \leftrightarrow x+y=0)$
(a) Is $\Re$ reflexive? ***(b)*** Is it symmetric? ***(c)*** Is it antisymmetric? ***(d)*** Is it transitive?

10. Same questions as above, but with: $x\Re y \leftrightarrow x-y\in\mathbb{Q}$
where $\mathbb{Q}$ denotes the set of rational numbers.

11. Same questions as above, but with: $x\Re y \leftrightarrow x=2y$

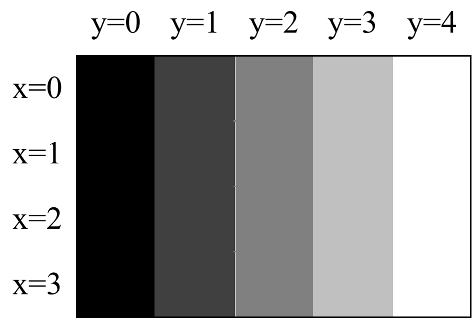
12. Same questions as above, but with: $x\Re y \leftrightarrow xy\geq 0$

13. Same questions as above, but with: $x\Re y \leftrightarrow x=1$

PART D. (5+3+2 marks)

In the field of image processing, an image can be defined in many different ways, and many binary relations are of interest. Here, an ***image*** is a total function from the set $(0..H-1)\times(0..W-1)$ to the set $0..2^{\ell}-1$, where H, W and ℓ are positive integers. It is an ***ℓ-bit greyscale image of height H and width W***. It can be represented by a 2-D array of numbers or of shades of grey. For example, the two arrays below represent the same 8-bit greyscale image I of height 4 and width 5; we have I(1,2)=128, I(3,1)=64, etc..

	y=0	y=1	y=2	y=3	y=4
x=0	0	64	128	192	255
x=1	0	64	128	192	255
x=2	0	64	128	192	255
x=3	0	64	128	192	255



An element of the codomain $0..2^{\ell}-1$ of an image is a ***grey level***. A total function from $0..2^{\ell}-1$ to $0..2^{\ell}-1$ is a ***grey-level mapping***, or ***lookup table***. A grey-level mapping can be used to change the grey levels of an image.

In the following, G denotes the set of bijections from $0..2^{\ell}-1$ to $0..2^{\ell}-1$ and $\mathcal{R}$ denotes the binary relation on the set of ℓ-bit greyscale images of height H and width W defined by:

$I \mathcal{R} J \leftrightarrow (\exists g \in G, \forall (x,y) \in (0..H-1)\times(0..W-1), J(x,y) = g(I(x,y)))$

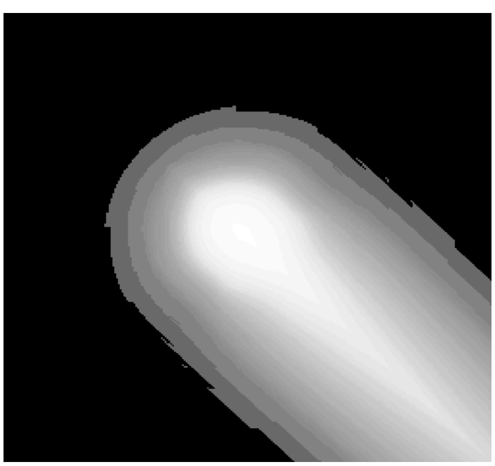
14. Show that $\mathcal{R}$ is an equivalence relation. (*Hint:* Use **A2** and **A3**.)

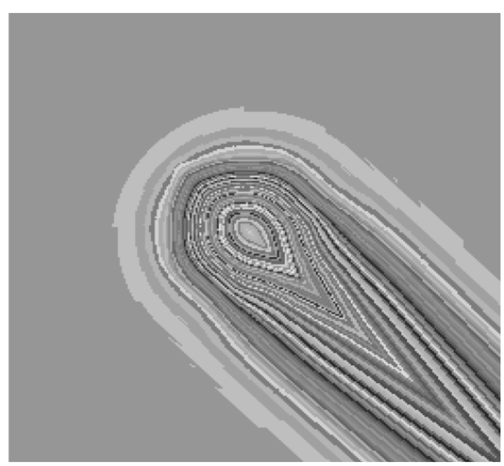
15. ***(a)*** Show that there exists an image whose equivalence class is of cardinality 2^{ℓ}.

(b) Show that if $H\times W \geq 2$ there is an image whose equivalence class is of cardinality $2^{\ell}\times(2^{\ell}-1)$.

(c) Show that if $H\times W \geq 2^{\ell}$ then there exists an image whose equivalence class is of cardinality $(2^{\ell})!$ (no formal proof required).

16. ***(a)*** Consider the images I (left) and J (right) below. J was built from I using some grey-level mapping g of G, i.e., J(x,y) was defined as g(I(x,y)), and the two images I and J are related by $\mathcal{R}$. How was g chosen?





(b) Consider the images I (left) and J (right) below. J was built from I using some grey-level mapping g of G, i.e., J(x,y) was defined as g(I(x,y)), and the two images I and J are related by $\mathcal{R}$. How was g chosen?