CIS1910 Discrete Structures in Computing (I)
Winter 2019, Solutions to Assignment 4

PART A.

1. ***(a)*** Consider two elements x_1 and x_2 of A such that $f(x_1)=f(x_2)$. Since $(x_1,f(x_1))$ and $(x_2,f(x_2))=(x_2,f(x_1))$ belong to F, the pairs $(f(x_1),x_1)$ and $(f(x_1),x_2)$ belong to F^{-1}. If f is injective then the relation f^{-1} is a function, which implies that $x_1=x_2$. ***(b)*** Consider two pairs (y,x_1) and (y,x_2) of F^{-1}. Then (x_1,y) and (x_2,y) belong to F, which means that $y=f(x_1)=f(x_2)$, which implies that $x_1=x_2$ according to the premise. We have shown that f^{-1} is a function, i.e., f is injective.

2. ***(a)*** Since f is a bijection, it is a function, and that function is injective, which means that f^{-1} is a ***function*** from B to A. ***(b)*** Consider an element y of B. Since f is surjective, there is an element x of A such that $(x,y)\in F$. Therefore, $(y,x)\in F^{-1}$, which means that y has an image under f^{-1}, i.e., y belongs to the domain of definition of f^{-1}. We have shown that f^{-1} is ***total***. ***(c)*** Consider an element x of A. Since f is total, there is an element y of B such that $(x,y)\in F$. Therefore, $(y,x)\in F^{-1}$, which means that x has a preimage under f^{-1}, i.e., x belongs to the range of f^{-1}. The function f^{-1} is ***surjective***. ***(d)*** Since $(F^{-1})^{-1}$ is equal to F, the relation $(f^{-1})^{-1}$ is equal to f, which we know is a function. We have shown that f^{-1} is ***injective***. ***(e)*** In the end, f^{-1} is ***bijective***. ***(f)*** Consider an element x of A. Since f is total, it is defined at x, and the pair $(x,f(x))$ belongs to F. Therefore, $(f(x),x)\in F^{-1}$, which means that $\mathbf{f^{-1}(f(x))=x}$.

3. ***(a)*** Consider an element x of A. Since f is total, x has an image f(x) under f, and that image belongs to B. Moreover, since g is total, f(x) has an image $g(f(x))=h(x)$ under g, and that image belongs to C. We have shown that h is ***total***. ***(b)*** Let z be an element of C. Since g is surjective, z has a preimage y under g, i.e., $g(y)=z$. Moreover, since f is surjective, y has a preimage x under f, i.e., $f(x)=y$. In the end, $g(f(x))=h(x)=z$, i.e., x is a preimage of z under h. The function h is ***surjective***. ***(c)*** Let x_1 and x_2 be two elements of A such that $h(x_1)=h(x_2)$, i.e., $g(f(x_1))=g(f(x_2))$. Since g is injective, $f(x_1)=f(x_2)$ (according to **A1a**), and since f is injective, $x_1=x_2$ (**A1a** again). Therefore, h is ***injective*** (according to **A1b**). ***(d)*** In the end, h is a ***bijection***.

PART B.

See Lab 1 Part B and Lab 7 Part B for the use of biconditionals.

4. ***(a)*** The domain of definition of f is the set of all the elements x of the domain ℝ such that 1/x belongs to the codomain ℝ. It is $\{x \in \mathbb{R} \mid 1/x \in \mathbb{R}\} = \{x \in \mathbb{R} \mid x \neq 0\} = \mathbb{R}^*$.

(b) If y=0 then the solution set is $\varnothing$.
If $y \neq 0$ we have $1/x=y \leftrightarrow x=1/y$ and the solution set is **{1/y}**.

(c) According to ***(b)***, any element y of the codomain of f has a preimage under f, except 0. The range of f is, therefore, $\mathbb{R}^*$.

(d) f is ***NOT total***, since its domain and domain of definition are not equal; see ***(a)***. f is ***NOT surjective***, since its codomain and range are not equal; see ***(c)***. Now, consider an element y of the codomain of f and two elements x_0 and x_1 of the domain. According to ***(b)***, y has at most one preimage under f: the number 1/y. Therefore, if (y,x_0) and (y,x_1) belong to the graph of the relation f^{-1}, then $x_0=x_1$. This means that f^{-1} is actually a function, and, therefore, f is ***injective***. Finally, f is ***NOT bijective***, since it is not total and not surjective.

(e) Let $I=\mathbb{R}^*$ and $J=\mathbb{R}^*$. Like f, the function $f_{(I,J)}$ is injective. Contrary to f, however, it is total (since its domain and domain of definition are equal) and surjective (since its codomain and range are equal). In the end, $f_{(I,J)}$ is bijective and its inverse is the function $y \mapsto 1/y$ from J to I. In other words (since we can choose the symbol x there instead of y), we have $\mathbf{f_{(I,J)}^{-1} = f_{(I,J)}}$.

5. ***(a)*** ℝ

(b) Let x and y be two real numbers. If y<0 then the solution set is $\varnothing$.
If y=0 then the solution set is **{0}**.
If y>0 we have $x^2=y \leftrightarrow (x=-\sqrt{y} \vee x=\sqrt{y})$ and the solution set is $\mathbf{\{-\sqrt{y},\sqrt{y}\}}$.

(c) According to ***(b)***, the range of f is $\mathbf{[0,+\infty[}$.

(d) f is ***total***. f is ***NOT surjective*** and, therefore, ***NOT bijective***. f is ***NOT injective*** either: for example, according to ***(b)***, the preimages of 1 under f are −1 and 1; since both (1,–1) and (1,1) belong to its graph, the relation f^{-1} is not a function.

(e) If $I=J=[0,+\infty[$ then the function $f_{(I,J)}$ is bijective and its inverse is:

$$\begin{aligned} &[0,+\infty[\to [0,+\infty[\\ &x \mapsto \sqrt{x} \end{aligned}$$

6. ***(a)*** **[0,+∞[**

(b) If $y<0$ then the solution set is ∅.
If $y\geq 0$ we have $\sqrt{x}=y \leftrightarrow x=y^2$ and the solution set is $\{y^2\}$.

(c) According to ***(b)***, the range of f is **[0,+∞[**.

(d) f is ***NOT total***, ***NOT surjective***, ***NOT bijective***, but it is ***injective***.

(e) If I=J=[0,+∞[then the function $f_{(I,J)}$ is bijective and its inverse is:

$$[0,+\infty[\ \to\ [0,+\infty[$$
$$x \mapsto x^2$$

7. ***(a)*** $\mathbb{R}$

(b) If $y<0$ then the solution set is ∅.
If $y=0$ then the solution set is **{0}**.
If $y>0$ we have $|x|=y \leftrightarrow (x=y \lor x=-y)$ and the solution set is $\{-y,y\}$.

(c) According to ***(b)***, the range of f is **[0,+∞[**.

(d) f is ***total***, but it is ***NOT surjective***, ***NOT injective***, ***NOT bijective***.

(e) If I=J=[0,+∞[then the function $f_{(I,J)}$ is bijective and its inverse is itself:

$$[0,+\infty[\ \to\ [0,+\infty[$$
$$x \mapsto x$$

8. ***(a)*** The domain of definition of f is the set of all the elements x
of the domain $\mathbb{R}$ such that $1/\sqrt{(x+1)}$ belongs to the codomain $\mathbb{R}$.
It is $\{x\in\mathbb{R} \mid 1/\sqrt{(x+1)}\in\mathbb{R}\} = \{x\in\mathbb{R} \mid x+1>0\} =]-1,+\infty[$.

(b) If $y\leq 0$ then the solution set is ∅. If $y>0$ then

$$1/\sqrt{(x+1)}=y$$
$$\leftrightarrow \sqrt{(x+1)}=1/y$$
$$\leftrightarrow x+1=1/y^2$$
$$\leftrightarrow x=-1+1/y^2$$

and the solution set is $\{-1+1/y^2\}$.

(c) The range of f is $\mathbb{R}^+$.

(d) f is ***NOT total***, ***NOT surjective***, ***NOT bijective***, but it is ***injective***.

(e) Let I=]−1,+∞[and J=]0,+∞[. The function $f_{(I,J)}$ is bijective and its inverse is:

$$]-1,+\infty[\ \to\ ]0,+\infty[$$
$$x \mapsto -1+1/x^2$$

PART C.

9. $x+y=0$

- Since $1+1\neq 0$, we have $1\not\mathfrak{R}1$.
  The proposition $\forall x, (x\mathfrak{R}x)$ is not true.
  The relation is ***NOT reflexive***.
- Consider any real numbers x and y. Assume $x\mathfrak{R}y$. Then $x+y=0$, i.e., $y+x=0$, i.e., $y\mathfrak{R}x$.
  Therefore, the proposition $\forall x, \forall y, (x\mathfrak{R}y \rightarrow y\mathfrak{R}x)$ is true.
  The relation is ***symmetric***.
- Since $1+(-1)=(-1)+1=0$, we have $1\mathfrak{R}-1$ and $-1\mathfrak{R}1$.
  The proposition $\forall x, \forall y, ((x\mathfrak{R}y \wedge y\mathfrak{R}x) \rightarrow x=y)$ is not true.
  The relation is ***NOT antisymmetric***.
- We have $1\mathfrak{R}-1$ and $-1\mathfrak{R}1$, but $1\not\mathfrak{R}1$.
  The proposition $\forall x, \forall y, \forall z, ((x\mathfrak{R}y \wedge y\mathfrak{R}z) \rightarrow x\mathfrak{R}z)$ is not true.
  The relation is ***NOT transitive***.

10. $x-y\in \mathbb{Q}$

- Consider any real number x. Since $x-x$, i.e., 0, is a rational number, we have $x\mathfrak{R}x$.
  The relation is ***reflexive***.
- Consider any real numbers x and y. Assume $x\mathfrak{R}y$. Then $x-y$ is a rational number (i.e., there exist two integers p and q such that $x-y=p/q$). Therefore, $y-x$ is a rational number (we have $y-x=P/Q$ with $P=-p$ and $Q=q$). In other words, $y\mathfrak{R}x$. The relation is ***symmetric***.
- Since $1-0$ and $0-1$ are rational numbers, we have $1\mathfrak{R}0$ and $0\mathfrak{R}1$.
  The relation is ***NOT antisymmetric***.
- Consider any real numbers x, y and z. Assume $x\mathfrak{R}y$ and $y\mathfrak{R}z$. Then $x-y$ and $y-z$ are rational numbers (say, p/q and p'/q'). Therefore, $x-z=(x-y)+(y-z)$ is a rational number too (we have $x-z=(p/q)+(p'/q')=(pq'+p'q)/(pq)=P/Q$ with $P=pq'+p'q$ and $Q=pq$). In other words, $x\mathfrak{R}z$.
  The relation is ***transitive***.

11. $x=2y$

- $1\not\mathfrak{R}1$. The relation is ***NOT reflexive***.
- $2\mathfrak{R}1$ but $1\not\mathfrak{R}2$. The relation is ***NOT symmetric***.
- Consider any real numbers x and y. Assume $x\mathfrak{R}y$ and $y\mathfrak{R}x$. Then $x=2y$ and $y=2x$. Therefore, $x=2(2x)=4x$ and $y=2(2y)=4y$, i.e., $x=0$ and $y=0$. Hence, $x=y$. The relation is ***antisymmetric***.
- We have $4\mathfrak{R}2$ and $2\mathfrak{R}1$, but $4\not\mathfrak{R}1$. The relation is ***NOT transitive***.

12. $xy\geq 0$

- Consider any real number x. Since $x^2\geq 0$, we have $x\mathfrak{R}x$. The relation is ***reflexive***.
- Consider any real numbers x and y. Assume $x\mathfrak{R}y$. Then $xy\geq 0$, i.e., $yx\geq 0$, i.e., $y\mathfrak{R}x$.
  The relation is ***symmetric***.
- We have $1\mathfrak{R}2$ and $2\mathfrak{R}1$. The relation is ***NOT antisymmetric***.
- We have $1\mathfrak{R}0$ and $0\mathfrak{R}-1$, but $1\not\mathfrak{R}-1$. The relation is ***NOT transitive***.

13. x=1

- Since $0\neq1$, we have $0 \not\mathfrak{R} 0$. The relation is ***NOT reflexive***.
- $1\mathfrak{R}2$ (since $1=1$). However, $2 \not\mathfrak{R} 1$ (since $2\neq1$). The relation is ***NOT symmetric***.
- Consider any real numbers x and y. Assume $x\mathfrak{R}y$ and $y\mathfrak{R}x$. Then x=1 and y=1. Therefore, x=y. The relation is ***antisymmetric***.
- Consider any real numbers x, y and z. Assume $x\mathfrak{R}y$ and $y\mathfrak{R}z$. Then x=1 (and y=1). Therefore, $x\mathfrak{R}z$. The relation is ***transitive***.

PART D.

14. Let I, J and K be ℓ-bit greyscale images of height H and width W.

(a) Consider the function $id : 0..2^{\ell}-1 \rightarrow 0..2^{\ell}-1$

$$u \mapsto u$$

id is a bijection, i.e., it is an element of G.
Moreover: $\forall(x,y)\in(0..H-1)\times(0..W-1)$, $I(x,y) = id(I(x,y))$
which means that $I \mathcal{R} I$. We have shown that $\mathcal{R}$ is ***reflexive***.

(b) Assume $I \mathcal{R} J$. Then, there exists an element g of G such that for any (x,y) of $(0..H-1)\times(0..W-1)$ we have $J(x,y) = g(I(x,y))$. We know from **A2** that g^{-1} is a bijection, i.e., it belongs to G. Moreover, according to **A2**, we have $g^{-1}(J(x,y)) = g^{-1}(g(I(x,y))) = I(x,y)$, which means that $J \mathcal{R} I$. We have shown that $\mathcal{R}$ is ***symmetric***.

(c) Assume $I \mathcal{R} J$ and $J \mathcal{R} K$. Then, there exist two elements g and h of G such that for any (x,y) of $(0..H-1)\times(0..W-1)$ we have $J(x,y) = g(I(x,y))$ and $K(x,y)=h(J(x,y))$, and, therefore, $K(x,y)=h(g(I(x,y)))$. Consider the function $k : 0..2^{\ell}-1 \rightarrow 0..2^{\ell}-1$

$$u \mapsto h(g(u))$$

We know from **A3** that k is a bijection, i.e., it belongs to G. In the end, we have found an element of G, the bijection k, such that for any (x,y) of $(0..H-1)\times(0..W-1)$ we have $K(x,y)=k(I(x,y))$. This means that $I \mathcal{R} K$. We have shown that $\mathcal{R}$ is ***transitive***.

(d) In the end, $\mathcal{R}$ is an ***equivalence relation***.

15. ***(a)*** Consider an image I whose range is {0}. Assume the image J is related to I. Then, there exists a bijection g of G such that for any (x,y) of $(0..H-1)\times(0..W-1)$ we have $J(x,y)=g(I(x,y))=g(0)$. Since there are 2^{ℓ} ways to choose g(0), there are 2^{ℓ} ways to choose J. The equivalence class of I is of cardinality 2^{ℓ}.

(b) Consider an image I whose range is {0,1}. Assume the image J is related to I. Then, there exists a bijection g of G such that for any (x,y) of $(0..H-1)\times(0..W-1)$ we have either $J(x,y)=g(I(x,y))=g(0)$ or $J(x,y)=g(I(x,y))=g(1)$. Since there are 2^{ℓ}

ways to choose g(0) and $2^\ell-1$ ways left to choose g(1), there are $2^\ell\times(2^\ell-1)$ ways to choose J. The equivalence class of I is of cardinality $2^\ell\times(2^\ell-1)$.

(c) Consider an image I whose range is $0..2^\ell-1$. Assume the image J is related to I. Using the same reasoning as above, we can show that there are $2^\ell\times(2^\ell-1)\times\ldots\times1$ ways to choose J. The equivalence class of I is of cardinality $(2^\ell)!$

16. ***(a)*** A random value (out of 2^ℓ) was chosen for g(0), a random value (out of the $2^\ell-1$ values left) was chosen for g(1), etc. ***(b)*** $g : u \mapsto (2^\ell-1)-u$