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CIS1910

# 2. Boolean Algebra

Reading assignment: Up to Section 2.1 of the zyBook



## N-ARY OPERATIONS

definitions and properties

## N-ARY OPERATIONS: Definitions 2.3

Consider a set S.

A function from S to S is also called a ***unary operation*** on S,
a function from $S^2$ to S is called a ***binary operation*** on S, etc.

$f : \mathbb{R} \to \mathbb{R}$
$u \mapsto u^2$ is a unary operation on $\mathbb{R}$.

$g : \mathbb{N}^2 \to \mathbb{N}$
$(u,v) \mapsto u^v$ is a binary operation on $\mathbb{N}$.

*From now on, when using the term "unary operation", or "binary operation", etc., we will always implicitly assume that the domain of definition of the operation is equal to its domain.*





## N-ARY OPERATIONS: Notations 2.4

Unary operations are often denoted by symbols like ~ instead of letters like f, and we then write ~u or $\tilde{u}$ instead of ~(u).

− (opposite) is a unary operation on $\mathbb{R}$.

Binary operations are often denoted by symbols like ★ instead of letters like f, and we then write u★v instead of ★((u,v)).

+ and × (addition and multiplication) are binary operations on $\mathbb{R}$.

## N-ARY OPERATIONS: Properties 2.5

Let ★ be a binary operation on a set S.

★ is ***idempotent*** iff for any u in S we have u★u=u.
★ is ***commutative*** iff for any (u,v) in $S^2$ we have u★v=v★u.
★ is ***associative*** iff for any (u,v,w) in $S^3$ we have u★(v★w)=(u★v)★w.

+ and × are commutative and associative.
*min* and *max* are idempotent, commutative and associative.

Let ★ and ❖ be two binary operations on a set S.

★ is ***distributive*** over ❖ iff for any (u,v,w) in $S^3$ we have
u★(v❖w)=(u★v)❖(u★w) and (v❖w)★u=(v★u)❖(w★u).

× is distributive over +.
*min* is distributive over *max* and vice versa.





## N-ARY OPERATIONS: Special Elements 2.6

Let ★ be a binary operation on a set S.

n∈S is a ***neutral element*** for ★ iff u★n=n★u=u for any u in S.
a∈S is an ***absorbing element*** for ★ iff u★a=a★u=a for any u in S.

0 is a neutral element for + and 1 is a neutral element for ×.
0 is an absorbing element for ×.

If m∈S and n∈S are neutral elements for ★ then m=n.
If a∈S and b∈S are absorbing elements for ★ then a=b.

## N-ARY OPERATIONS: Precedence Rules 2.7

Let ~ be a unary operation on a set S,
let ★ and ❖ be two binary operations on S,
and let u, v and w be three elements of S.

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The expression ~u★v does not make sense, unless:

~ is given ***higher precedence*** than ★ ~u★v=(~u)★v
OR ~ is given ***lower precedence*** than ★ ~u★v=~(u★v)

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The expression u★v❖w does not make sense, unless:

★ is given higher precedence than ❖ u★v❖w=(u★v)❖w
OR ★ is given lower precedence than ❖ u★v❖w=u★(v❖w)





## N-ARY OPERATIONS: Associativity Rules 2.8

Let ~ be a unary operation on a set S,
let ★ and ❖ be two binary operations on S,
and let u, v and w be three elements of S.
Assume ~, ★ and ❖ have the ***same precedence***.

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The expression ~u★v does not make sense, unless:

***left-to-right associativity*** is assumed ~u★v=(~u)★v
OR ***right-to-left associativity*** is assumed ~u★v=~(u★v)

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The expression u★v★w does not make sense, unless:

★ is associative u★v★w=(u★v)★w=u★(v★w)
OR left-to-right associativity is assumed u★v★w=(u★v)★w
OR right-to-left associativity is assumed u★v★w=u★(v★w)

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The expression u★v❖w does not make sense, unless:

left-to-right associativity is assumed u★v❖w=(u★v)❖w
OR right-to-left associativity is assumed u★v❖w=u★(v❖w)

n-ary operations

## DEFINITIONS AND PROPERTIES



## DEFINITION 2.10

Consider a 4-tuple (B,+,·,¯) where B is a set,
+ and · are binary operations on B
and ¯ is a unary operation on B.

(B,+,·,¯) is a ***Boolean algebra***
iff there exist two distinct elements
0 and 1 of B such that:

| | |
|---|---|
| 0 is the neutral element for + | 1 is the neutral element for · |
| + is commutative | · is commutative |
| + is associative | · is associative |
| + is distributive over · | · is distributive over + |
| For any element u of B: $u+\bar{u}=1$ | For any element u of B: $u\cdot\bar{u}=0$ |

## EXAMPLE 2.11

Consider the 4-tuple $(\{0,1\},+,\cdot,\overline{\ })$ where
$+$ is the binary operation on $\{0,1\}$ defined by

$0+0 = 0$ and $0+1 = 1+0 = 1+1 = 1$

$\cdot$ is the binary operation on $\{0,1\}$ defined by

$1\cdot 1 = 1$ and $1\cdot 0 = 0\cdot 1 = 0\cdot 0 = 0$

$\overline{\ }$ is the unary operation on $\{0,1\}$ defined by

$\overline{0} = 1$ and $\overline{1} = 0$

$(\{0,1\},+,\cdot,\overline{\ })$ is a Boolean algebra.





## FUNDAMENTAL LAWS 2.12

Let $(B,+,\cdot,\overline{\ })$ be a Boolean algebra
Let 0 be the neutral element for +
Let 1 be the neutral element for ·
For all u, v, w of B:

**** identity laws*** $u+0=u$, $u\cdot 1=u$

***commutative laws*** $u+v=v+u$, $u\cdot v=v\cdot u$

***associative laws*** $(u+v)+w=u+(v+w)$, $(u\cdot v)\cdot w=u\cdot(v\cdot w)$

**** distributive laws*** $u+(v\cdot w)=(u+v)\cdot(u+w)$, $u\cdot(v+w)=(u\cdot v)+(u\cdot w)$

**** complement laws*** $u+\overline{u}=1$, $u\cdot\overline{u}=0$

$0+0=0$ and $0+1=1+0=1+1=1$
$1\cdot 1=1$ and $1\cdot 0=0\cdot 1=0\cdot 0=0$

$\overline{0}=1$ and $\overline{1}=0$

***law of the double complement*** $\overline{\overline{u}}=u$

**** domination laws*** $u+1=1$, $u\cdot 0=0$

***idempotent laws*** $u+u=u$, $u\cdot u=u$

***De Morgan's laws*** $\overline{u+v}=\overline{u}\cdot\overline{v}$, $\overline{u\cdot v}=\overline{u}+\overline{v}$

## BASIC TERMINOLOGY 2.13

Let (B,+,·,¯) be a Boolean algebra
Let 0 be the neutral element for +
Let 1 be the neutral element for ·

\+ is the ***Boolean sum***.

· is the ***Boolean product***.

¯ is the ***Boolean complementation***.

0 is the ***zero element***.

1 is the ***unit element***.

A ***Boolean value*** is an element of B.

A ***Boolean variable*** is a variable that represents an element of B.

An n-ary ***Boolean operation***, or ***Boolean function*** of degree n, is an n-ary operation on B.





## LITERALS, MINTERMS, MAXTERMS 2.14

Let (B,+,·,¯) be a Boolean algebra with zero element 0, unit element 1.

A ***literal*** is a Boolean variable or its complement.

A ***minterm*** of degree $n \in \mathbb{N}^*$ is the product of n literals, with exactly one literal per variable.

A ***maxterm*** of degree $n \in \mathbb{N}^*$ is the sum of n literals, with exactly one literal per variable.

u and $\bar{u}$ are minterms of degree 1; $u+\bar{v}$ a maxterm of degree 2;
$\bar{u} \cdot v \cdot \bar{w}$ a minterm of degree 3; 0, $\bar{u}+u$, $\overline{u \cdot v}$ not minterms nor maxterms.

## BOOLEAN EXPRESSIONS 2.15

Let $(B,+,\cdot,\overline{\phantom{x}})$ be a Boolean algebra with zero element 0, unit element 1.

A ***Boolean expression*** is a finite sequence of symbols such that:

- ☐ each symbol is a bracket or denotes a Boolean value, a Boolean variable, or a Boolean operation;
- ☐ the sequence makes sense, i.e., it results in a Boolean value once Boolean values are assigned to the Boolean variables.

Two Boolean expressions are ***equivalent*** if the two resulting values are equal whatever the values assigned to the variables.

$u+\overline{v}$, $u\cdot v$, $\overline{(1+\overline{u})}\cdot v$ are Boolean expressions; $u\overline{v}+$, $u+\cdot v$, $)\overline{u}+1(v$ are not.
$\overline{\overline{u}\cdot\overline{v}}$ and $u+v$ are equivalent since $\overline{\overline{u}\cdot\overline{v}} = u+v$ for any u and v in B.





## DUALITY PRINCIPLE 2.16

If $(B,+,\cdot,\overline{\phantom{x}})$ is a Boolean algebra with zero element 0 and unit element 1,
$(B,\cdot,+,\overline{\phantom{x}})$ is a Boolean algebra with zero element 1 and unit element 0.

Consider a Boolean expression that does not involve:
Boolean operations other than $+$, $\cdot$ and $\overline{\phantom{x}}$;
Boolean values other than 0 and 1.

Its ***dual*** is the Boolean expression obtained by interchanging $+$ and $\cdot$, and 0 and 1.

0 and 1 are dual; $\overline{u}+v$ and $\overline{u}\cdot v$ are dual; $u\cdot(\overline{\overline{v}+w})$ and $u+(\overline{\overline{v}\cdot w})$ are dual.

Consider two Boolean expressions as above.
If the two expressions are equivalent,
the dual expressions are equivalent.

***duality principle***

n-ary operations
definitions and properties

END