CIS1910

3. Logic

- ☐ The basis of all mathematical reasoning and of all automated reasoning
- ☐ Practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages...

Reading assignment: Up to Section 3.1 of the zyBook

PROPOSITIONAL LOGIC

predicate logic

PROPOSITIONAL LOGIC: Propositions 3.3

A ***proposition*** is a declarative sentence,
i.e., it is a sentence that declares a fact.

"Toronto is the capital of France"
"It will rain tomorrow"
"1+1=2"

This fact is either true or false, i.e.,
the ***truth value*** of the proposition is either ***T*** or ***F***.

A ***propositional variable*** is a variable that represents an element of the set of all propositions.
It is denoted by a letter like p, q, r.

PROPOSITIONAL LOGIC: Operations 3.4

A ***propositional operation*** is an operation on the set of all propositions. The most common operations are:

- ¬ ("not", ***negation***, unary)
- ∧ ("and", ***conjunction***, binary)
- ∨ ("or", ***disjunction***, binary)
- → ("if... then...", ***conditional***, binary)
- ↔ ("if and only if", ***biconditional***, binary)

A propositional operation is defined by a ***truth table***. For example:

p	¬p
F	T
T	F

p	q	p∧q
F	F	F
F	T	F
T	F	F
T	T	T

p	q	p∨q
F	F	F
F	T	T
T	F	T
T	T	T

p	q	p→q
F	F	T
F	T	T
T	F	F
T	T	T

p	q	p↔q
F	F	T
F	T	F
T	F	F
T	T	T

PROPOSITIONAL LOGIC: Expressions 3.5

A ***propositional expression*** is a finite sequence of symbols.
The accepted symbols are:

T (which denotes a proposition that is true),
F (which denotes a proposition that is false),
p, q, r, etc. (which denote propositional variables),
¬, ∧, ∨, →, ↔, etc. (which denote propositional operations),
and brackets.

The sequence should make sense, i.e.,
it should become a proposition once specific propositions are considered.
Note that a truth table can be attached to any propositional expression.

T, p, p∧F, (¬p)∨q, (q→(¬r))↔p
are propositional expressions.
The table attached to (¬p)∨q is:

p	q	¬p	(¬p)∨q
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

PROPOSITIONAL LOGIC: Equivalences 3.6

Two propositional expressions are ***equivalent***
iff they always have the same truth value.
We then use the symbol ≡

A ***tautology*** is a propositional expression
that is always true, i.e., that is equivalent to T.

A ***contradiction*** is a propositional expression
that is always false, i.e., that is equivalent to F.

A ***contingency*** is a propositional expression
that is neither a tautology nor a contradiction.

p→q ≡ (¬q)→(¬p)
p→q ≡ (¬p)∨q
p↔q ≡ (p→q)∧(q→p)

¬p∨p is a tautology.
¬p∧p is a contradiction.

PROPOSITIONAL LOGIC: A Note on the Conditional 3.7

$p \to q$: there are different ways to express it in English.

"if p, then q"	"q if p"	"p is sufficient for q"
"if p, q"	"q when p"	"a sufficient condition for q is p"
"p implies q"	"q unless not p"	"q is necessary for p"
"p only if q"	"q follows from p"	"a necessary condition for p is q"

In $p \to q$, p is the ***antecedent*** and q is the ***consequent***

$q \to p$ is the ***converse*** of $p \to q$
$(\neg p) \to (\neg q)$ is the ***inverse*** of $p \to q$
$(\neg q) \to (\neg p)$ is the ***contrapositive*** of $p \to q$

PROPOSITIONAL LOGIC: The Link to Boolean Algebra 3.8

Let $\mathscr{P}$ be the set of all propositions.
$(\mathscr{P}, \vee, \wedge, \neg)$ behaves like a Boolean algebra:

$$p \vee F \equiv p \qquad p \vee \neg p \equiv T$$
$$p \wedge T \equiv p \qquad p \wedge \neg p \equiv F$$

$$p \vee q \equiv q \vee p \qquad (p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$p \wedge q \equiv q \wedge p \qquad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Therefore, the idempotent laws, the De Morgan's laws, etc., apply.

propositional logic

PREDICATE LOGIC

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PREDICATE LOGIC: Predicates (1/2) 3.10

Let U be the set of all cities in the world.
Let u be an element of U.
Let P(u) be the statement "u *is the capital of France*".

P(u) becomes a proposition
once a specific element is considered for u.

P(*Paris*) is the proposition "*Paris is the capital of France*" (and it is true).
P(*Toronto*) is the proposition "*Toronto is the capital of France*" (it is false).

U is the ***universe*** of the variable u.
P is a unary ***predicate***:

$$\begin{aligned} P : U &\to \mathscr{P} \\ u &\mapsto P(u) \end{aligned}$$

PREDICATE LOGIC: Predicates (2/2) 3.11

Consider $Q : \mathbb{N}\times\mathbb{Z}\rightarrow \mathscr{P}$
$(u,v)\mapsto Q(u,v)$ where $Q(u,v)$ is the statement "$u+v=0$".

$Q(u,v)$ becomes a proposition
once specific elements are considered for u and v.

$Q(0,1)$ is the proposition "$0+1=0$" (and it is false).
$Q(1,-1)$ is the proposition "$1+(-1)=0$" (and it is true).

Q is a binary predicate.
The universe of u is $\mathbb{N}$.
The universe of v is $\mathbb{Z}$.

PREDICATE LOGIC: Universal Quantifier (1/2) 3.12

Consider $P : \{-1,0,1\}\rightarrow \mathscr{P}$
$u \mapsto P(u)$ where $P(u)$ is the statement "$u^2\geq u$".

$\forall u, P(u)$ (read "for all u, P of u") is the proposition $P(-1)\wedge P(0)\wedge P(1)$.
It is true, since $P(-1)$, $P(0)$ and $P(1)$ are all true,
i.e., since $P(u)$ is true for all possible values of u.

The proposition $\forall u, P(u)$ is the ***universal quantification*** of P.
The symbol $\forall$ is the ***universal quantifier***.

Note
You can consider a subset U′ of {−1,0,1} and the proposition $\forall u\in U', P(u)$.
Whatever the predicate P, the proposition $\forall u\in \{\}, P(u)$ *is true.*

PREDICATE LOGIC: Universal Quantifier (2/2) 3.13

Consider $Q : \mathbb{R} \to \mathscr{P}$
$u \mapsto Q(u)$ where $Q(u)$ is the statement "$u^2 \geq u$".

$\forall u, Q(u)$ is false, since there is a u for which $Q(u)$ is false.
$Q(0.5)$, for instance, is false: 0.5 is a ***counterexample*** of $\forall u, Q(u)$.

Note
The proposition $\forall u \in [1,+\infty[, Q(u)$ *can also be written as* $\forall u \geq 1, Q(u)$ *(and it is true).*

PREDICATE LOGIC: Existential Quantifier (1/2) 3.14

Consider $P : \{-1,0,1\} \to \mathscr{P}$
$u \mapsto P(u)$ where $P(u)$ is the statement "$|u| > u$".

$\exists u, P(u)$ (read "there exists u such that P of u")
is the proposition $P(-1) \vee P(0) \vee P(1)$.
It is true, since, e.g., $P(-1)$ is true.

The proposition $\exists u, P(u)$ is the ***existential quantification*** of P.
The symbol $\exists$ is the ***existential quantifier***.

Note
You can consider a subset U′ of {−1,0,1} and the proposition $\exists u \in U', P(u)$.
Whatever the predicate P, the proposition $\exists u \in \{\}, P(u)$ *is false.*

PREDICATE LOGIC: Existential Quantifier (2/2) 3.15

Consider $Q : \mathbb{R} \to \mathscr{P}$, $u \mapsto Q(u)$ where $Q(u)$ is the statement "$|u| > u$".

$\exists u, Q(u)$ is true, since there is a u for which $Q(u)$ is true.
$Q(-1)$, for instance, is true: -1 is an ***example*** of $\exists u, Q(u)$.

Note
The proposition $\exists u \in [0, +\infty[, Q(u)$ can also be written as $\exists u \geq 0, Q(u)$ (and it is false).

PREDICATE LOGIC: Operations on Predicates 3.16

Consider the predicates $P : U \to \mathscr{P}$, $u \mapsto P(u)$ and $Q : U \to \mathscr{P}$, $u \mapsto Q(u)$

$\neg P$ denotes the predicate $\neg P : U \to \mathscr{P}$, $u \mapsto \neg(P(u))$

$P \wedge Q$ denotes the predicate $P \wedge Q : U \to \mathscr{P}$, $u \mapsto P(u) \wedge Q(u)$

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$\exists u, \neg P(u)$ denotes the proposition $\exists u, (\neg P)(u)$.

$\forall u, (P(u) \wedge Q(u))$ denotes the proposition $\forall u, (P \wedge Q)(u)$.

......

PREDICATE LOGIC: Nested Quantifiers 3.17

Consider the predicate $P : U \times V \to \mathscr{P}$
$(u,v) \mapsto P(u,v)$

Consider the predicate $Q : U \to \mathscr{P}$
$u \mapsto \forall v,\ P(u,v)$

$\forall u,\ Q(u)$ can be rewritten as $\forall u,\ (\forall v,\ P(u,v))$
$\exists u,\ Q(u)$ can be rewritten as $\exists u,\ (\forall v,\ P(u,v))$

Consider the predicate $R : U \to \mathscr{P}$
$u \mapsto \exists v,\ P(u,v)$

$\forall u,\ R(u)$ can be rewritten as $\forall u,\ (\exists v,\ P(u,v))$
$\exists u,\ R(u)$ can be rewritten as $\exists u,\ (\exists v,\ P(u,v))$

PREDICATE LOGIC: Expressions 3.18

A ***predicate expression*** is a finite sequence of symbols.
The accepted symbols are symbols that denote:

- quantifiers (e.g., $\forall$, $\exists$),
- predicates (e.g., P, Q, R),
- variables (e.g., u, v, w),
- values (e.g., "John", 2, {1})
- operations (e.g., $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$),
- commas and brackets.

The sequence should make sense, i.e.,
it should become a proposition once specific predicates are considered.

$\exists u,\ P(u)$	$\exists u,\ (\forall v,\ P(u,v))$
$\forall u,\ (P(u) \wedge \neg Q(u))$	$\exists u,\ [(\forall v,\ P(u,v)) \vee Q(u)]$
$(\exists u,\ \neg P(u)) \rightarrow (\forall u,\ P(u))$	$[\exists u,\ (\forall v,\ P(u,v))] \wedge (\forall u,\ Q(u))$

PREDICATE LOGIC: Scope and Binding 3.19

The part of a predicate expression to which a quantifier is applied is the ***scope*** of the quantifier.

When a quantifier is used on a variable,
it ***binds*** the variable in the scope of the quantifier.

scope of   scope of

(∃u, ¬P(u))→(∀u, P(u))

(∃v, ¬P(v))→(∀u, P(u))
(∃v, ¬P(v))→(∀v, P(u))

NO!

(∃v, ¬P(v))→(∀u, P(v))

scope of

∃u, [(∀v, P(u,v))∨Q(u)]

scope of

NO!

∃u, [(∀v, P(u,v))∨Q(v)]
∃u, [(∀u, P(u,u))∨Q(u)]

NO!

∃v, [(∀u, P(v,u))∨Q(v)]

PREDICATE LOGIC: Equivalences 3.20

Two predicate expressions are ***equivalent***
iff they always yield the same truth value.
We then use the symbol ≡

¬(∀u, P(u)) ≡ ∃u, ¬P(u)
¬(∃u, P(u)) ≡ ∀u, ¬P(u)

∀u, (P(u)∧Q(u)) ≡ (∀u, P(u))∧(∀u, Q(u))
∃u, (P(u)∨Q(u)) ≡ (∃u, P(u))∨(∃u, Q(u))

∀u, (∀v, P(u,v)) ≡ ∀v, (∀u, P(u,v)) ≡ ∀(u,v), P(u,v)
∃u, (∃v, P(u,v)) ≡ ∃v, (∃u, P(u,v)) ≡ ∃(u,v), P(u,v)

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END