5. Set Theory

CARDINALITY OF A SET 5.2

Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, then S is a ***finite set***.
n is the ***cardinality*** of S. We write |S|=n.

A set which is not finite is an ***infinite set***.

A={0,1}, B={0,1,2,3,…,99},
C={1,1/2,1/3,1/4,…}, D={Pascal,Guelph,3.1,B}
A, B, D, {}, {{}} are finite: |A|=2, |B|=100, |D|=4, |{}|=0, |{{}}|=1
C is infinite.

POWER SET OF A SET 5.3

The ***power set*** of a set S is the set $\mathbf{2^S}$ of all subsets of S.

$2^{\{0,1\}} = \{\emptyset,\{0\},\{1\},\{0,1\}\}$, $2^{\emptyset} = \{\emptyset\}$, $2^{\{\emptyset\}} = \{\emptyset,\{\emptyset\}\}$.
Let S be a set: we have $\emptyset \in 2^S$ and $S \in 2^S$.

Assume S is a finite set: we have $|2^S|=2^{|S|}$.

TRUTH SET OF A UNARY PREDICATE 5.4

$\{u \in \mathcal{U} \mid P(u)\}$, where P is a unary predicate $P : \mathcal{U} \to \mathscr{P}$
$u \mapsto P(u)$,
denotes the set of elements of $\mathcal{U}$ such that P(u) is true.
It is defined using ***set builder notation***. It is called the ***truth set*** of P.
It is a subset of $\mathcal{U}$, and $\mathcal{U}$ is called the ***universal set***.

$P : \mathbb{Z} \to \mathscr{P}$
$n \mapsto$ "n is even"

The truth set of P is
$\{n \in \mathbb{Z} \mid$ "n is even" $\} = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}, n=2k\} = \{\dots,-2,0,2,4,\dots\}$

TRUTH SET OF A BINARY PREDICATE 5.5

$\{(u,v) \in \mathcal{U} \times \mathcal{V} \mid Q(u,v)\}$, where Q is a binary predicate $Q : \mathcal{U} \times \mathcal{V} \to \mathscr{P}$
$(u,v) \mapsto Q(u,v)$,
denotes the set of elements of $\mathcal{U} \times \mathcal{V}$ such that Q(u,v) is true.
It is defined using ***set builder notation***. It is called the ***truth set*** of Q.
It is a subset of $\mathcal{U} \times \mathcal{V}$, and $\mathcal{U} \times \mathcal{V}$ is called the ***universal set***.

$Q : \mathbb{Z}^2 \to \mathscr{P}$
$(m,n) \mapsto$ "mn=1"

The truth set of Q is $\{(m,n) \in \mathbb{Z}^2 \mid mn=1\} = \{(-1,-1),(1,1)\}$

COMMON NUMBER SETS 5.6

$\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n>0\}$
$2\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}, n=2k\} = \{2k\}_{k \in \mathbb{Z}}$
$\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$
$\mathbb{Q} = \{x \in \mathbb{R} \mid \exists p \in \mathbb{Z}, \exists q \in \mathbb{Z}^+, x=p/q\} = \{p/q\}_{(p,q) \in \mathbb{Z} x \mathbb{Z}^+}$
$\mathbb{Q}^- = \{x \in \mathbb{Q} \mid x<0\}$

NUMBER INTERVALS 5.7

Examples of Integer Intervals
Let m and n be two integers.

m..n = $\{x\in\mathbb{Z} \mid m\leq x \land x\leq n\}$ ***bounded***
m..+∞ = $\{x\in\mathbb{Z} \mid m\leq x\}$ ***half-bounded***
−∞..n = $\{x\in\mathbb{Z} \mid x\leq n\}$ ***left-unbounded***

Examples of Real Intervals
Let u and v be two real numbers.

[u,v] = $\{x\in\mathbb{R} \mid u\leq x \land x\leq v\}$ ***closed, bounded***
]u,v[= $\{x\in\mathbb{R} \mid u< x \land x< v\}$ ***open, bounded***
[u,v[= $\{x\in\mathbb{R} \mid u\leq x \land x< v\}$ ***half-closed, right-open, bounded***
[u,+∞[= $\{x\in\mathbb{R} \mid u\leq x\}$ ***closed, left-bounded***
]−∞,v[= $\{x\in\mathbb{R} \mid x< v\}$ ***open, half-unbounded***

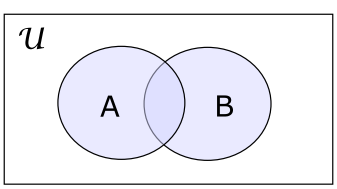
UNION, INTERSECTION, COMPLEMENT 5.8

Let $\mathcal{U}$ be the universal set (any nonempty set), and let A and B be two sets (i.e., two subsets of $\mathcal{U}$).

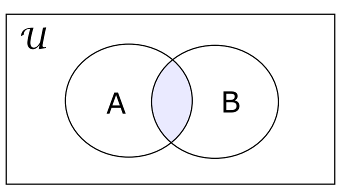
The set $\{x\in\mathcal{U} \mid x\in A \lor x\in B\}$ is called the ***union*** of the sets A and B. It is denoted by $A\cup B$. The symbol $\cup$ defines a binary operation on $2^{\mathcal{U}}$.

The set $\{x\in\mathcal{U} \mid x\in A \land x\in B\}$ is called the ***intersection*** of A and B. It is denoted by $A\cap B$. The symbol $\cap$ defines a binary operation on $2^{\mathcal{U}}$. (Note that if $A\cap B=\{\}$, then we say that A and B are ***disjoint***.)

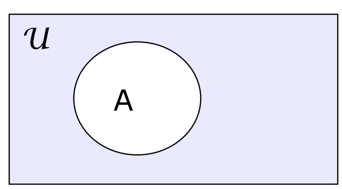
The set $\{x\in\mathcal{U} \mid x\notin A\}$ is called the ***complement*** of the set A. It is denoted by $\overline{A}$. The symbol $\overline{}$ defines a unary operation on $2^{\mathcal{U}}$.



$A\cup B$



$A\cap B$



$\overline{A}$

THE LINK TO BOOLEAN ALGEBRA 5.9

Theorem

$(2^{\mathcal{U}}, \cup, \cap, \overline{})$ is a Boolean algebra:
we have, for all A, B and C in $2^{\mathcal{U}}$,

$$A \cup \{\} = A \qquad A \cup \overline{A} = \mathcal{U}$$
$$A \cap \mathcal{U} = A \qquad A \cap \overline{A} = \{\}$$

$$A \cup B = B \cup A \qquad (A \cup B) \cup C = A \cup (B \cup C)$$
$$A \cap B = B \cap A \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

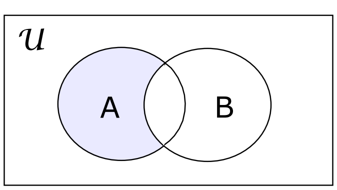
Therefore, the idempotent laws, the De Morgan's laws, etc., apply.

DIFFERENCE AND SYMMETRIC DIFFERENCE 5.10

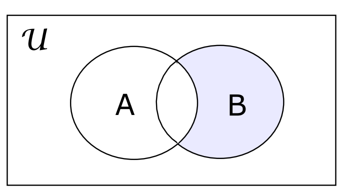
Let $\mathcal{U}$ be the universal set (any nonempty set),
and let A and B be two sets (i.e., two subsets of $\mathcal{U}$).

The set $\{x \in \mathcal{U} \mid x \in A \wedge x \notin B\}$ is called the ***difference*** of A and B.
It is denoted by A**−**B. Note that $A - B = A \cap \overline{B}$ and, usually, $A - B \neq B - A$.

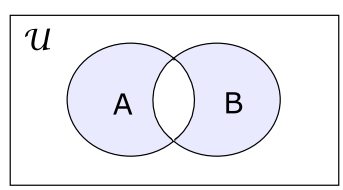
The set $\{x \in \mathcal{U} \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$
is the ***symmetric difference*** of the sets A and B.
It is denoted by A**+**B. Note that $A + B = (A - B) \cup (B - A)$, and $A + B = B + A$.



A−B



B−A



A+B

MEMBERSHIP TABLES 5.11

A	B	A∪B
0	0	0
0	1	1
1	0	1
1	1	1

Consider any element x of the universal set $\mathcal{U}$:

If x does not belong to A but belongs to B then x belongs to A∪B.
If x belongs to A and belongs to B then x belongs to A∪B.

A	B	A∩B
0	0	0
0	1	0
1	0	0
1	1	1

A	$\overline{A}$
0	1
1	0

Reading assignment: Up to Section 6.1 of the zyBook

5.12

END