CIS1910

6. Binary Relations

Reading assignment: Up to Section 6.1 of the zyBook

OVER TWO SETS

on a set

equivalence relations

order relations

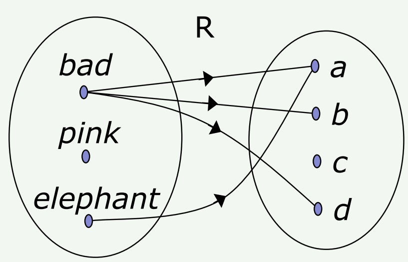
OVER TWO SETS: Definition 6.3

Let U and V be two sets. A ***binary relation over*** U ***and*** V is a triple R=(U,V,G) where G is a subset of UxV.

U is the ***domain***, V is the ***codomain***, G is the ***graph***.
When (u,v)∈G we write **uRv** (read "u is related to v by R").

U={*bad*,*pink*,*elephant*}, V={*a*,*b*,*c*,*d*},
G={(*bad*,*a*),(*bad*,*b*),(*bad*,*d*),(*elephant*,*a*)}, R=(U,V,G)

The relation R can be read "is a word that contains the letter".
We have, e.g., *bad* R *d*, *elephant* R *a*, *elephant* $\not R$ *b*, *pink* $\not R$ *a*



R	*a*	*b*	*c*	*d*
bad	1	1	0	1
pink	0	0	0	0
elephant	1	0	0	0

OVER TWO SETS: Representation 6.4

Let U and V be two sets. A ***binary relation over*** U ***and*** V is a triple R=(U,V,G) where G is a subset of UxV.

U is the ***domain***, V is the ***codomain***, G is the ***graph***.
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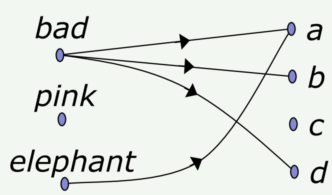
U={*bad*,*pink*,*elephant*}, V={*a*,*b*,*c*,*d*},
G={(*bad*,*a*),(*bad*,*b*),(*bad*,*d*),(*elephant*,*a*)}, R=(U,V,G)

digraph representation

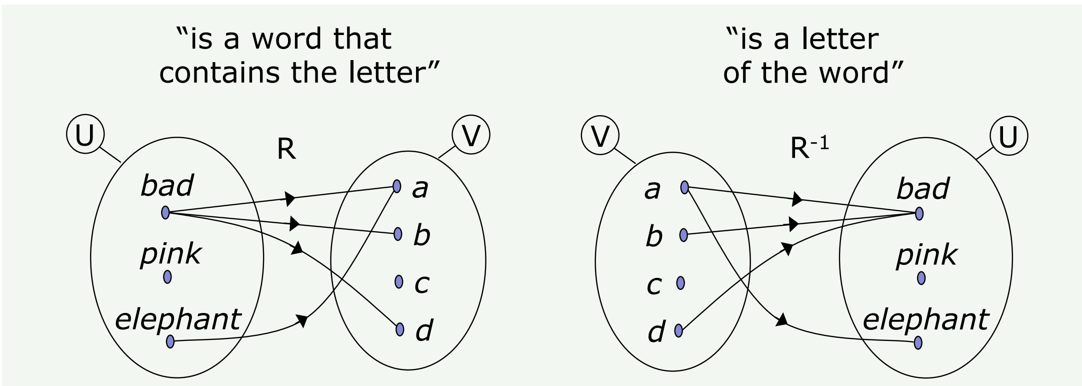


matrix representation

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

OVER TWO SETS: Inverse 6.5

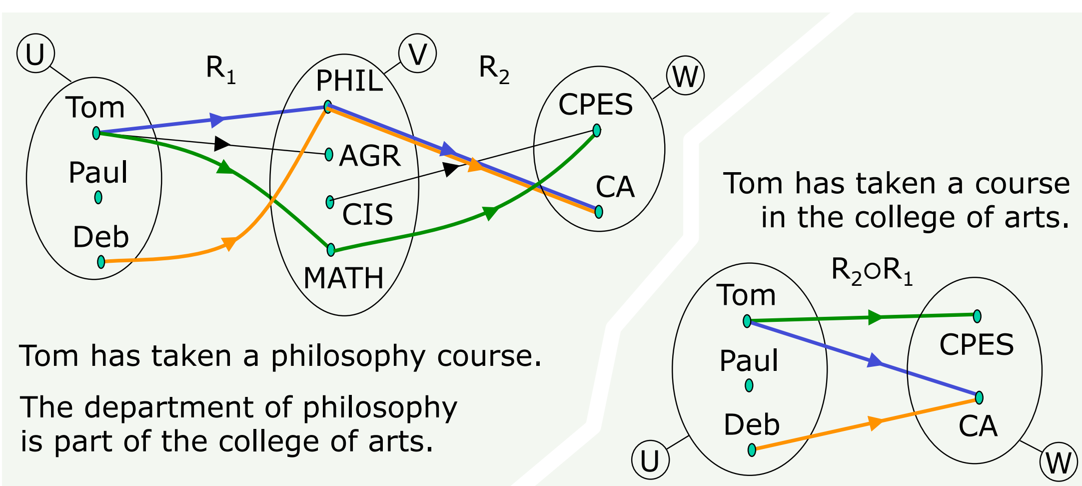
Consider a binary relation R=(U,V,G).
The ***inverse*** of R is **R^{-1}**$=(V,U,G^{-1})$
with $G^{-1}=\{(v,u)\in VxU \mid uRv\}$.



Reading assignment: Up to Section 6.3 of the zyBook

OVER TWO SETS: Composition 6.6

Consider two binary relations $R_1=(U,V,G_1)$ and $R_2=(V,W,G_2)$.
The ***composition*** of R_1 and R_2 is $R_2 \mathbf{o} R_1=(U,W,G)$
(read "R_2 compose R_1") with $G=\{(u,w)\in UxW \mid \exists v\in V, (uR_1v \wedge vR_2w)\}$.



Reading assignment: Up to Section 6.4 of the zyBook over two sets

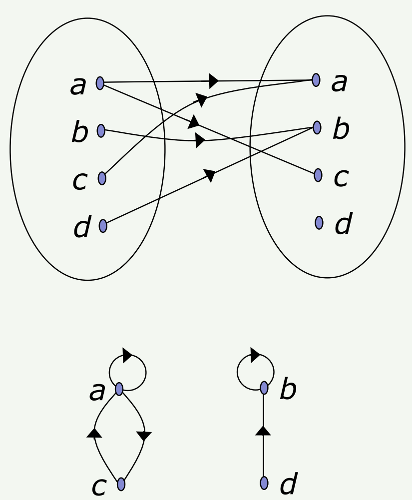
ON A SET

equivalence relations

order relations

ON A SET: Definition 6.8

A relation over U and U is a ***binary relation on*** U.



	a	b	c	d
a	1	0	1	0
b	0	1	0	0
c	1	0	0	0
d	0	1	0	0

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

ON A SET: Properties 6.9

Let R be a binary relation on some set U.

R is ***reflexive*** iff: $\forall u, (uRu)$
R is ***symmetric*** iff: $\forall u, \forall v, (uRv \rightarrow vRu)$
R is ***antisymmetric*** iff: $\forall u, \forall v, ((uRv \land vRu) \rightarrow u=v)$
R is ***transitive*** iff: $\forall u, \forall v, \forall w, ((uRv \land vRw) \rightarrow uRw)$

ON A SET: Closures (1/2) 6.10

Consider a binary relation on a set.

	a	b	c	d
a	0	0	1	0
b	0	1	0	0
c	0	0	0	0
d	1	0	0	1

reflexive closure

Add to the graph of the relation as few elements as possible such that the resulting relation is **reflexive**.

transitive closure

Add to the graph of the relation as few elements as possible such that the resulting relation is **transitive**.

symmetric closure

Add to the graph of the relation as few elements as possible such that the resulting relation is **symmetric**.

ON A SET: Closures (2/2) 6.11

Consider a binary relation on a set.

	a	b	c	d
a	0	0	1	0
b	0	1	0	0
c	0	0	0	0
d	1	0	0	1

reflexive closure

	a	b	c	d
a	**1**	0	1	0
b	0	1	0	0
c	0	0	**1**	0
d	1	0	0	1

symmetric closure

	a	b	c	d
a	0	0	1	**1**
b	0	1	0	0
c	**1**	0	0	0
d	1	0	0	1

transitive closure

	a	b	c	d
a	0	0	1	0
b	0	1	0	0
c	0	0	0	0
d	1	0	**1**	1

Reading assignment: Up to Section 6.4 of the zyBook

over two sets

on a set

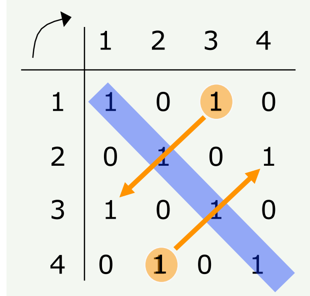
EQUIVALENCE RELATIONS

order relations

EQUIVALENCE RELATIONS: Definition 6.13

Let R be a binary relation on some set U.
We say that R is an ***equivalence relation*** on U iff R is **reflexive**, **symmetric** and **transitive**

i.e. $\forall u, (uRu)$
$\forall u, \forall v, (uRv \rightarrow vRu)$
$\forall u, \forall v, \forall w, ((uRv \wedge vRw) \rightarrow uRw)$



equivalence relation

EQUIVALENCE RELATIONS: Notation 6.14

An equivalence relation is often denoted by a symbol that resembles =
(which is an equivalence relation on the set of real numbers)
e.g.
=, ≡, ~

u ~ v reads "u *is equivalent to* v"
or "u *and* v *are equivalent*"

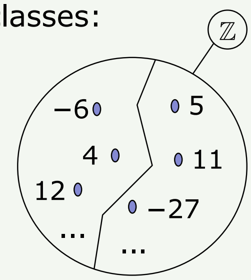
EQUIVALENCE RELATIONS: Equivalence Classes 6.15

Let ~ be an equivalence relation on a set U and let u be an element of U.

The ***equivalence class*** of u (with respect to ~) is **[u]**={v∈U | u~v}.
Any element v of [u] is called a ***representative*** of [u], and [u]=[v].

Consider the binary relation on ℤ defined by: for any x and y in ℤ, x is related to y iff x and y have the same parity. This relation is an equivalence relation. There are two equivalence classes:

[0]=[2]=[−2]=2ℤ={…,−4,−2,0,2,4,…}
[1]=[3]=[−1]=ℤ−2ℤ={…,−3,−1,1,3,5,…}



EQUIVALENCE RELATIONS: Partitions 6.16

Let ~ be an equivalence relation on a set U and let u be an element of U.

Any equivalence class is a nonempty set.
Any two distinct equivalence classes are disjoint.
Any element of U belongs to some equivalence class.

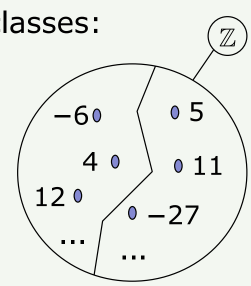
} *the equivalence classes form a* ***partition*** *of* U

Consider the binary relation on ℤ defined by: for any x and y in ℤ, x is related to y iff x and y have the same parity. This relation is an equivalence relation. There are two equivalence classes:

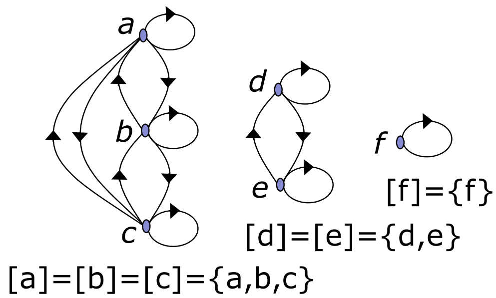
[0]=[2]=[−2]=2ℤ={…,−4,−2,0,2,4,…}
[1]=[3]=[−1]=ℤ−2ℤ={…,−3,−1,1,3,5,…}

We have: [0]≠{} and [1]≠{}
[0]∩[1]={}
[0]∪[1]=ℤ

} {[0],[1]} *is a 2-partition of* ℤ



EQUIVALENCE RELATIONS: Example 6.17



The digraph above represents an equivalence relation on {a,b,c,d,e,f}.
There are three equivalence classes: [a], [d] and [f].

[a]≠{} and [d]≠{} and [f]≠{}
[a]∩[d]={} and [a]∩[f]={} and [d]∩[f]={}
[a]∪[d]∪[f]={a,b,c,d,e,f}

} {[a], [d], [f]} *is a 3-partition of* {a,b,c,d,e,f}

over two sets
on a set
equivalence relations

ORDER RELATIONS

ORDER RELATIONS: Definition 6.19

Let R be a binary relation on some set U.
We say that R is an ***order relation*** on U
iff R is **reflexive**, **antisymmetric** and **transitive**.

i.e. $\forall u, (uRu)$
$\forall u, \forall v, ((uRv \land vRu) \rightarrow u=v)$
$\forall u, \forall v, \forall w, ((uRv \land vRw) \rightarrow uRw)$

(U,R) is then called an ***ordered set***.

R	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	1	0	1	0
4	1	1	0	1

order relation

ORDER RELATIONS: Total 6.20

Let R be an order relation on some set U.

Two elements u and v of U are ***comparable*** iff uRv or vRu.
Two elements of U are ***incomparable*** iff they are not comparable.

If any two elements are comparable,
then R is a ***total order relation*** on U
and (U,R) is a ***totally ordered set***.

R	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	1	0	1	0
4	1	1	0	1

order relation
(not total)

ORDER RELATIONS: Notation 6.21

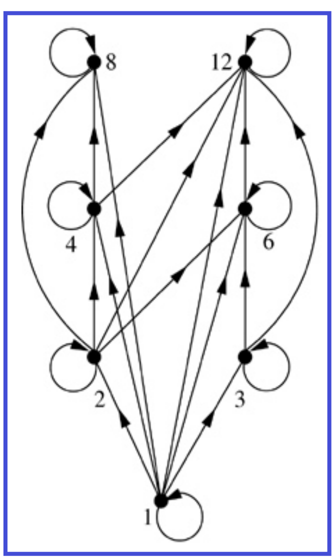
An order relation is often denoted by a symbol that resembles ≤
(which is an order relation on the set of real numbers)
e.g.,
≤, ⪯

u ⪯ v reads "u *is less than or equal to* v"
or "v *is greater than or equal to* u"

ORDER RELATIONS: Hasse Diagram (1/5) 6.22

An order relation can be represented by a ***Hasse diagram***

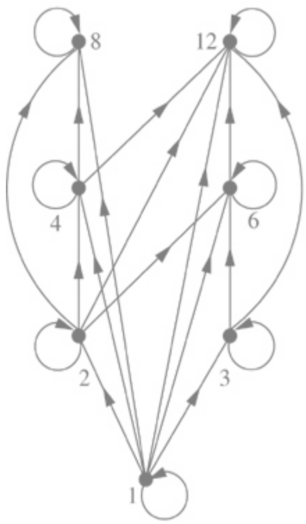
1/ represent the relation by a digraph

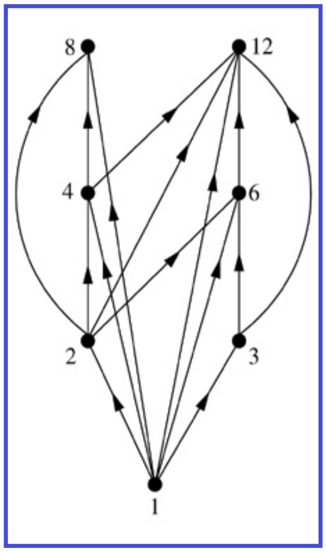


ORDER RELATIONS: Hasse Diagram (2/5) 6.23

An order relation can be represented by a ***Hasse diagram***

2/ remove the loops

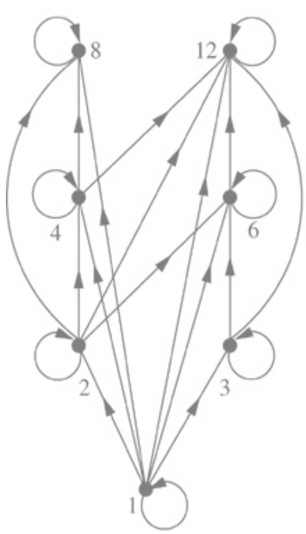


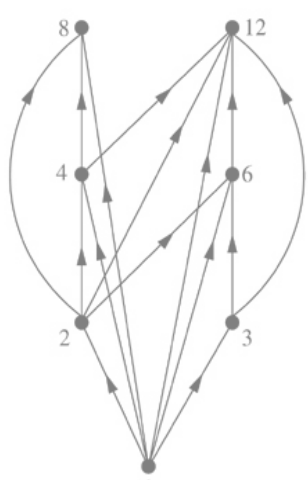


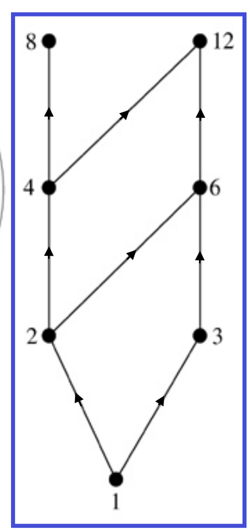
ORDER RELATIONS: Hasse Diagram (3/5) 6.24

An order relation can be represented by a ***Hasse diagram***

3/ remove the edges that can be retrieved from transitivity



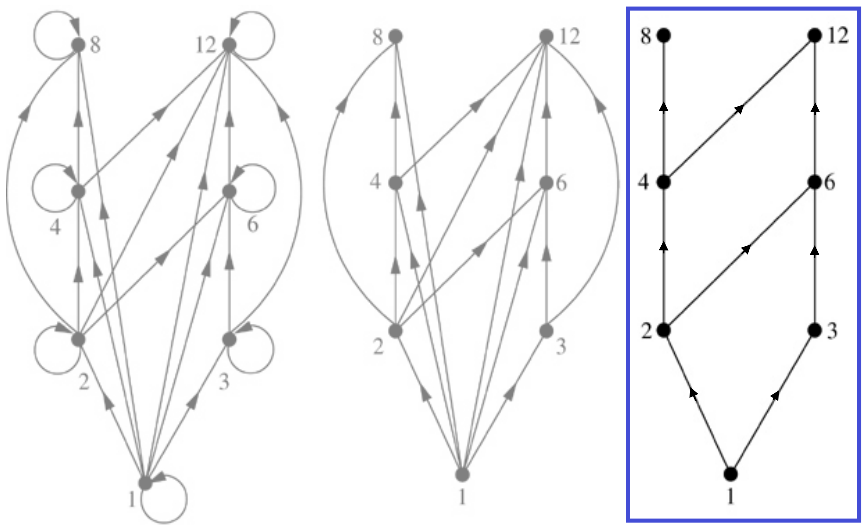




ORDER RELATIONS: Hasse Diagram (4/5) 6.25

An order relation can be represented by a ***Hasse diagram***

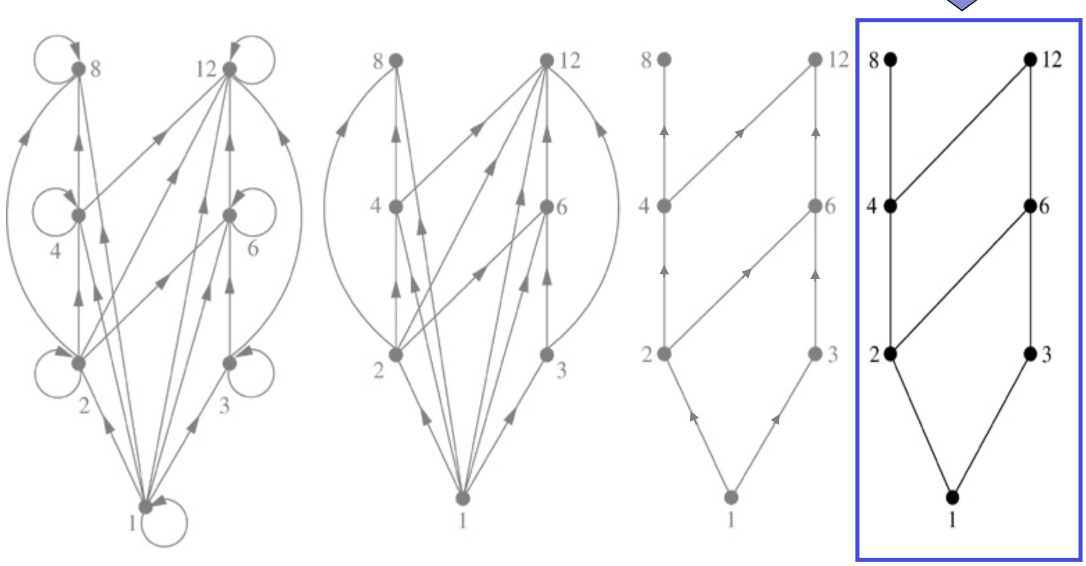
4/ arrange the vertices so that the initial vertex of each edge is below the terminal vertex



ORDER RELATIONS: Hasse Diagram (5/5) 6.26

An order relation can be represented by a ***Hasse diagram***

5/ remove all the arrows on the directed edges



ORDER RELATIONS: Maxima and Minima 6.27

Consider an ordered set $(U, \preceq)$.

Let e be an element of U.

e is ***maximal*** iff: $\forall u \in U,\ (e \preceq u \rightarrow e = u)$

e is ***minimal*** iff: $\forall u \in U,\ (u \preceq e \rightarrow u = e)$

e is the* ***greatest element*** iff: $\forall u \in U,\ (u \preceq e)$

e is the* ***least element*** iff: $\forall u \in U,\ (e \preceq u)$

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ORDER RELATIONS: Upper and Lower Bounds 6.28

Consider an ordered set $(U, \preceq)$.

Let V be a subset of U and let e be an element of U.

e is an ***upper bound*** of V iff: $\forall v \in V,\ (v \preceq e)$

e is a ***lower bound*** of V iff: $\forall v \in V,\ (e \preceq v)$

e is the* ***supremum*** of V iff
e is an upper bound of V and $e \preceq u$ for any upper bound u of V.

e is the* ***infimum*** of V iff
e is a lower bound of V and $u \preceq e$ for any lower bound u of V.

Reading assignment: Up to Section 7.2 of the zyBook

over two sets

on a set

equivalence relations

order relations

END