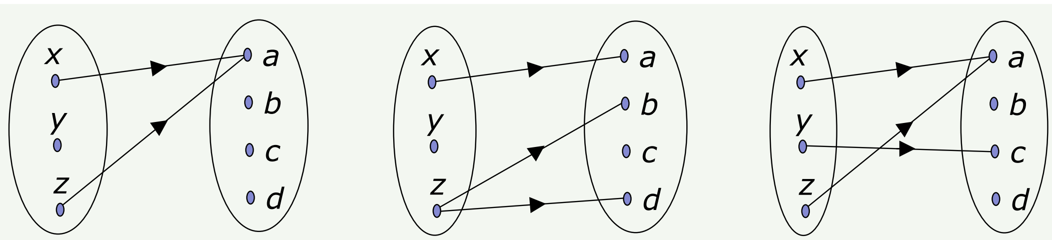
7. Another Look at Functions

FUNCTIONS VS. RELATIONS 7.2

Let $\mathcal{R}$ be a binary relation over two sets U and V.
$\mathcal{R}$ is a ***function*** iff: $\forall u \in U, \forall (v_1,v_2) \in V^2, [(u\mathcal{R}v_1 \wedge u\mathcal{R}v_2) \rightarrow v_1=v_2]$

Then:
- ☐ its ***domain of definition*** is $\{u \in U \mid \exists v \in V, u\mathcal{R}v\}$
- ☐ its ***range*** is $\{v \in V \mid \exists u \in U, u\mathcal{R}v\}$
- ☐ the letter **f** (or **g**, **h**, …) is preferred to $\mathcal{R}$
- ☐ the notation **f(u)=v** is preferred to u f v
- ☐ **SEE SLIDES 1.15-1.22**



function not a function function

IMAGES AND PREIMAGES 7.3

Let f be a function from U to V. Assume U′⊆U and V′⊆V.

f is ***defined on*** U′ iff U′ is a subset of its domain of definition.

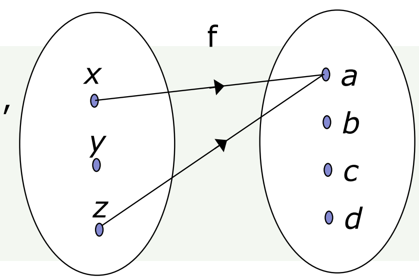
The ***image*** of U′ (under f) is the set **f(U′)**={v∈V | ∃u∈U′, f(u)=v}.
Note that the range of f is f(U).

The ***preimage*** of V′ is the set **f^{-1}(V′)**={u∈U | ∃v∈V′, f(u)=v}.
Note that the domain of definition of f is f^{-1}(V).

f(y)
f^{-1}(a)
f^{-1}(b)

f(x)=f(z)=a,
f({x})=f({z})={a}, f({y})={},
f({x,y})=f({x,y,z})={a}

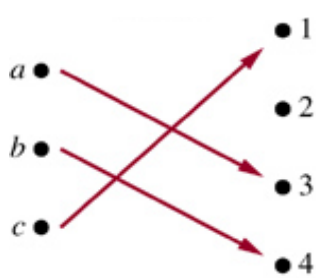
f^{-1}({a})=f^{-1}({a,b,c,d})={x,z},
f^{-1}({b})=f^{-1}({b,c,d})={}



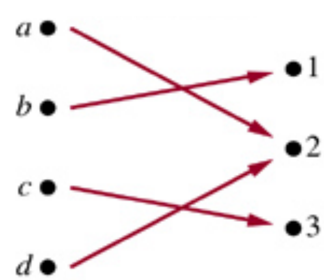
PROPERTIES (and first examples) 7.4

Let f be a function from U to V.

- f is ***total*** iff its domain of definition is U.
- f is ***surjective*** (*onto*) iff its range is V.
- f is ***injective*** (*one-to-one*) iff its inverse f^{-1} is a function.
- f is ***bijective*** (*one-to-one correspondence*) iff f is total, surjective, and injective.



total,
injective

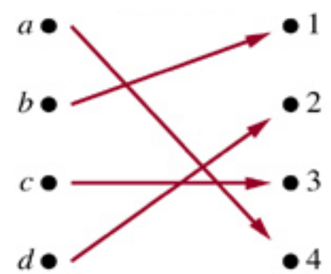


total,
surjective

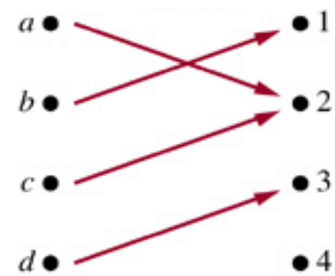
PROPERTIES (and other examples) 7.5

Let f be a function from U to V.

- f is ***total*** iff its domain of definition is U.
- f is ***surjective*** (*onto*) iff its range is V.
- f is ***injective*** (*one-to-one*) iff its inverse f^{-1} is a function.
- f is ***bijective*** (*one-to-one correspondence*) iff f is total, surjective, and injective.



total, ***surjective***, ***injective***, ***bijective***



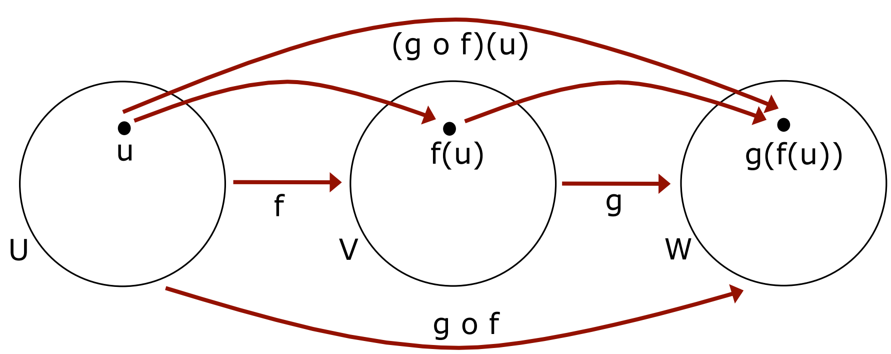
total

COMPOSITION 7.6

Consider two functions f : U → V and g : V → W.
The composition $g \circ f$ of f and g is a function: $g \circ f : U \to W$

$$u \mapsto g(f(u))$$

$f : \mathbb{R} \to \mathbb{R}$ $u \mapsto 3u - 1$	$g : \mathbb{R} \to \mathbb{R}^{+}$ $u \mapsto 4u^2 + 1$	$g \circ f : \mathbb{R} \to \mathbb{R}^{+}$ $u \mapsto 4(3u-1)^2 + 1$	~~$f \circ g$~~

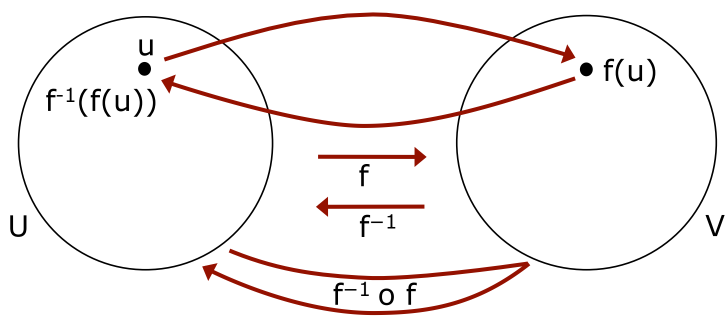


COMPOSITION (with the inverse) 7.7

Consider a bijective function $f : U \to U$.
Then f^{-1} is a bijective function too, $f \circ f^{-1} : U \to U$, $u \mapsto u$ and $f^{-1} \circ f : U \to U$, $u \mapsto u$

identity function on U, usually denoted by $\mathbf{Id_U}$

$f : \mathbb{R} \to \mathbb{R}$	$f^{-1} : \mathbb{R} \to \mathbb{R}$
$u \mapsto 3u - 1$	$u \mapsto (u+1)/3$



MONOTONIC FUNCTIONS 7.8

A ***function of an integer variable*** (resp. ***real variable***) is a function whose domain is a subset of $\mathbb{Z}$ (resp. $\mathbb{R}$).

An ***integer function*** (resp. a ***real function***) is a function whose codomain is a subset of $\mathbb{Z}$ (resp. $\mathbb{R}$).

Let f be a real function of a real variable defined on a set S:

- ☐ f is ***increasing*** on S iff $\forall (u,v) \in S^2, [u<v \to f(u)<f(v)]$
- ☐ f is ***decreasing*** on S iff $\forall (u,v) \in S^2, [u<v \to f(u)>f(v)]$
- ☐ f is ***nondecreasing*** on S iff $\forall (u,v) \in S^2, [u \le v \to f(u) \le f(v)]$
- ☐ f is ***nonincreasing*** on S iff $\forall (u,v) \in S^2, [u \le v \to f(u) \ge f(v)]$
- ☐ f is ***monotonic*** on S iff f is one of the above.

$f : \mathbb{R} \to \mathbb{R}$	$g : \mathbb{R} \to \mathbb{R}$	$h : \mathbb{R} \to \mathbb{R}$
$x \mapsto x - 1$	$x \mapsto 1/(x-1)$	$x \mapsto (x-1)^2$
increasing on $\mathbb{R}$	decreasing on $]-\infty,1[$ decreasing on $]1,+\infty[$	decreasing on $]-\infty,1]$ increasing on $[1,+\infty[$

OPERATIONS ON FUNCTIONS 7.9

Consider two functions f and g from U to V.
Let ~ be a unary operation on V and let ★ be a binary operation on V.

$$\mathbf{\sim f} : U \to V,\quad u \mapsto \sim(f(u)) \qquad \mathbf{f \star g} : U \to V,\quad u \mapsto f(u) \star g(u)$$

Consider two real functions f and g from U to V.

$$\sqrt{\mathbf{f}} : U \to V,\quad u \mapsto \sqrt{f(u)} \qquad \mathbf{f^2} : U \to V,\quad u \mapsto [f(u)]^2 \qquad \frac{\mathbf{1}}{\mathbf{f}} : U \to V,\quad u \mapsto \frac{1}{f(u)} \qquad |\mathbf{f}| : U \to V,\quad u \mapsto |f(u)|$$

$$\mathbf{f + g} : U \to V,\quad u \mapsto f(u) + g(u) \qquad \mathbf{fg} : U \to V,\quad u \mapsto f(u)g(u) \qquad \mathbf{f - g} : U \to V,\quad u \mapsto f(u) - g(u)$$

Reading assignment: Up to Section 7.4 of the zyBook

7.10

END