A general approach to the fuzzy modeling of spatial relationships

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Abstract. How to satisfactorily model spatial relationships between 2D or 3D objects? If the objects are far enough from each other, they can be approximated by their centers. If they are not too far, not too close, they can be approximated by their minimum bounding rectangles or boxes. If they are close, no such simplifying approximation should be made. Two concepts are at the core of the approach described in this paper: the concept of the $\mathcal{F}$-histogram and that of the $\mathcal{F}$-template. The basis of the former was laid a decade ago; since then, it has naturally evolved and matured. The latter is much newer, and has dual characteristics. Our aim here is to present a snapshot of these concepts and of their applications. It is to highlight (and reflect on) their duality—a duality that calls for a clear distinction between the terms spatial relationship, relationship to a reference object, and relative position. Finally, it is to identify directions for future research.

1 Introduction

Philosophers, physicists and mathematicians have been debating about space for centuries. Here, space is considered Euclidean and independent of time (our apology to Einstein). It is not, however, a mere abstract void (and Leibniz would rejoice): talking about space implies talking about (spatial) objects and relationships. Indeed, space is viewed as "the structure defined by the set of spatial relationships between objects" [58]. In the present paper as in related literature, space is usually two- or three-dimensional, with a Cartesian coordinate system. A physical object is of no interest in itself; the focus is on the part of space it occupies. Objects, therefore, are seen as subsets of the Euclidean space. A point, a line segment, a disk, a toroid, the union of these, are examples of objects. An object may be bounded or unbounded, convex or concave, open or closed, connected or disconnected, etc. Practically, it is either in raster or vector form. A raster object in 2D space, for example, is sometimes seen as the union of unit squares (pixels), and others as a cloud of points (pixel centers). Finally, note that fuzzy subsets of the Euclidean space may also be considered. Fuzzy sets make it possible to encapsulate information regarding the imprecision or the uncertainty in the spatial extent of some physical objects. There is, in the end, a variety of spatial objects.

So there is a variety of spatial relationships. Some are language-based, in the sense that they are naturally referred to using everyday terms, e.g., the relationships "right" (is to the right of), "far" (is far from), "touch" (touches). Others are math-based, and may or may not be named (e.g., the 512 relationships defined by the 9-intersection model). Some are binary; they involve two objects only (e.g., object *A* is to the right of object *B*). Others are not (e.g., object *A* is between objects *B* and *C*). In this paper, we limit our discussion to binary relationships, which are by far the most common subject of studies. They are usually categorized into directional (e.g., "right"), distance (e.g., "far"), and topological (e.g., "touch") relationships. This is not surprising, since angles and distances are at the core of Euclidean geometry, and Euclidean spaces are, above all, topological spaces. Other categories, however, are sometimes considered (e.g., "intersect" is set-theoretical before being topological).

Spatial relationships are often modeled by fuzzy relations on the set of all objects. Consider, for example, the statement "*A* is far from *B*". In many everyday situations, one would find it neither completely true nor completely false (even if *A* and *B* are very simple crisp objects). A fuzzy model of "far" attaches a numerical value to the pair (A,B), and this value is seen as the degree of truth of the statement above. Not only the use of fuzzy relations seems more natural than the use of standard, all-or-nothing relations, but it also allows two fundamental questions to be answered. How to identify the most salient relationship between two given objects in a scene? How to identify the object that best satisfies a given relationship to a reference object? Answering these questions comes down to calculating and comparing the degrees of truth of several statements. See Figs. 1 and 2.

These statements, however, are not independent from each other. Part of the calculation of each degree of truth might therefore be common to all degrees of truth and yield an intermediate result, interesting if only for efficiency purposes. This result can be seen as a quantitative representation of either the relative position between the two objects (first question, Fig. 1) or the relationship to the reference object (second question, Fig. 2). What we argue here is that a clear distinction should be made between the terms spatial relationship (a binary relationship), relationship to a reference object (a unary relationship), and relative position. True, the position of an object with respect to another may be described in terms of spatial relationships. However, it may also have a representation of its own, as mentioned above. Ideally, such a representation should allow any relationship between the two objects to be assessed. Practically, this is never the case. The information relative to a given relationship might not have been captured by the representation, or might have been encapsulated in an unfathomable way. One representation may be better suited than another to the assessment of some relationships, and vice versa. At any rate, we may be interested in relative positions for what they are, and not in any particular relationship (e.g., when carrying out a scene matching task).

Section 2 illustrates the discussion above. Its aim is to clarify, through examples, the differences between the terms spatial relationship, relationship to a reference object, and relative position. Sections 3 and 4 introduce the two concepts at the core of the general approach described in this paper, while pointing out dual characteristics. Sections 5 and 6 show how these concepts may rely on two others, also with dual characteristics. Section 7 deals with algorithmic issues. Many applications have been studied; Section 8 presents a review of the related literature. Finally, directions for future research are given in Section 9. Note that spatial relationships have been studied for many years, in many disciplines, including cognitive science, linguistics, geography and artificial intelligence. See, e.g., [7] [8] [13] [14] [18] [19] [39] [40]. The approach described here focuses on fuzzy models of spatial relationships (as opposed to, e.g., qualitative models) and is general only in the sense that: a variety of spatial objects can be handled (e.g., crisp or fuzzy, connected or disconnected, in raster form or in vector form); a variety of spatial information can be captured and exploited (i.e., directional, distance, topological); there is a variety of current and potential applications.

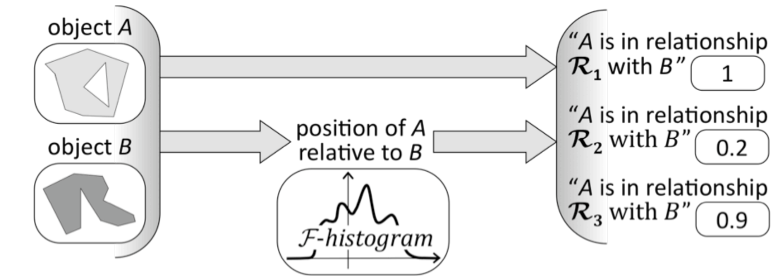


Fig. 1 How to identify the most salient relationship between two given objects A and B? Here, A and B are represented by vector data, the position of A relative to B is represented by an $\mathcal{F}$-histogram (Section 3), and the 3 statements by fuzzy logic values. The answer to the question is $\mathcal{R}_1$.

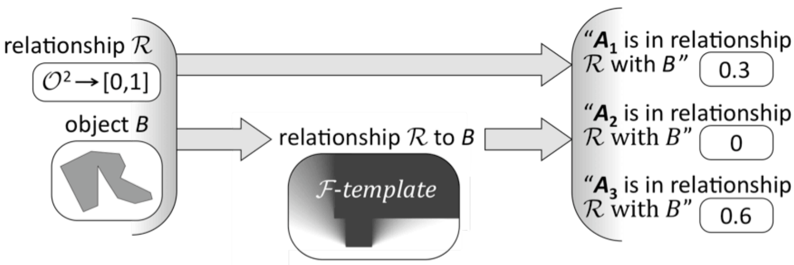


Fig. 2 How to identify the object that best satisfies a given relationship $\mathcal{R}$ to a reference object B? Here, $\mathcal{R}$ is represented by a fuzzy binary relation, B by vector data, the relationship $\mathcal{R}$ to B by an $\mathcal{F}$-template (Section 4), and the 3 statements by fuzzy logic values. The answer to the question is A_3.

2 An important distinction

2.1 Relative position vs. relationship—Example

Consider two points p and q in the 2D space. Possible representations of the position of p relative to q are the tuple (x_p,y_p,x_q,y_q), whose elements are the Cartesian coordinates of p and q; the pair (x_{qp},y_{qp}), whose elements are the Cartesian coordinates of the vector qp; the pair (ρ_{qp},θ_{qp}), whose elements are the polar coordinates of qp; the angle θ_{qp}; etc. The first representation is trivial. The second one, (x_{qp},y_{qp}), is much more interesting. Although some information about p and q is lost, there is no loss of information about the position of p relative to q (assuming that relative position is invariant to translation). The third representation has the same characteristic. However, it is better suited for the assessment of distance relationships. These relationships cannot be assessed from the fourth representation, θ_{qp}. Too much information is lost. Nonetheless, θ_{qp} is a very compact representation, well suited for the assessment of directional relationships. For example, assuming that angular coordinates belong to $(-\pi,\pi]$ and that the polar axis is horizontal and pointing to the right, we may consider that the degree of truth of the statement "p is to the right of q" is $\min\{1, \max\{0,1-2|\theta_{qp}|/\pi\}\}$. In other words, the fuzzy relation $\mathcal{R}$ defined by

$$\mathcal{R}(p,q) = \min\left\{1,\max\left\{0,1-\frac{2}{\pi}\left|\theta_{qp}\right|\right\}\right\} \tag{1}$$

is a fuzzy model of the *binary directional relationship* "right". If $\theta_{qp}=0$ then $\mathcal{R}(p,q)=1$, i.e., p is definitely to the right of q. If $\theta_{qp}=\pi/2$ then $\mathcal{R}(p,q)=0$, i.e., p is definitely not to the right of q. In the end, once the *relative position* θ_{qp} has been calculated given the Cartesian coordinates of p and q (a painful task if you are using only pen and paper), the statements "p is to the right of q", "p is above q", "p is in direction 45° of q", etc., can be evaluated with comparatively much less effort. The link with Fig. 1 should now be clear to the reader. Note that this example can be easily adapted to the 3D case.

2.2 Relationship vs. relationship to a reference object—Example

Here, an object is a "friendly" set of points in a rectangular region R of the 2D space, i.e., it is a nonempty, bounded, connected, regular closed set of points, included in R. Consider two objects A and B. Let $|B|$ be the area of B. For any two points p and q, let $|qp|$ be the distance between p and q.

We call

$$d(A,B) = \inf_{p\in A,\, q\in B} |qp| \tag{2}$$

the distance between A and B, and we call

$$s(B) = 2\sqrt{\frac{|B|}{\pi}} \tag{3}$$

the size of B (it is the diameter of a disk whose area is $|B|$). The fuzzy relation $\mathcal{R}$ defined by

$$\mathcal{R}(A,B) = \max\left\{0, 1 - \frac{d(A,B)}{s(B)\, t}\right\}, \tag{4}$$

where t denotes a positive real number, is a fuzzy model of the *binary distance relationship* "close". If the distance between A and B is 0, then $\mathcal{R}(A,B)=1$, i.e., A is definitely close to B. If the distance between A and B is t times larger than B, then $\mathcal{R}(A,B)=0$, i.e., A is not close at all to B. Now, given B, assume we are asked to evaluate the statement "A is close to B" for a large number of objects A. Going through (2) and (4) every time would be inefficient. A better strategy is to compute the function d_B defined on R by

$$d_B(p) = \inf_{q\in B} |qp| \tag{5}$$

and then use the fact that

$$d(A,B) = \inf_{p\in A} d_B(p)\,. \tag{6}$$

Or, compute the function

$$\overline{d}_B(p) = \max\left\{0, 1 - \frac{d_B(p)}{s(B)\, t}\right\} \tag{7}$$

and use the fact that

$$\mathcal{R}(A,B) = \sup_{p\in A} \overline{d}_B(p)\,. \tag{8}$$

Once $\overline{d}_B$ has been computed, the statements "A_1 is close to B", "A_2 is close to B", etc., can be readily evaluated. d_B and $\overline{d}_B$ are two possible representations of the *unary distance relationship* "close to B". See the link with Fig. 2. Note that d_B is usually known as a distance map. Again, this example can be easily adapted to the 3D case.

3 $\mathcal{F}$-histograms

One of the two concepts at the core of the general approach described in this paper is the concept of the $\mathcal{F}$-histogram. Its basis was laid a decade ago [23]. Since then, of course, the concept has evolved and matured. The idea and assumption behind it are that acceptable representations of relative positions can be obtained by reducing the handling of all 3D and 2D objects to the handling of 1D entities.

Notation and terminology are as follows. S is the Euclidean space. A *direction* θ is a unit vector. $\theta^{\perp}$ is the subspace orthogonal to θ that passes through the *origin* ω (an arbitrary point of S). The expression $S_\theta(p)$ denotes the line in direction θ that passes through the point p. Now, consider a fuzzy subset A of S. The membership degree of p in A is $\mu_A(p)$. For any $\alpha \in [0,1]$, the α-cut of A is $A^\alpha=\{p \in S \mid \mu_A(p) \geq \alpha\}$. The (fuzzy) intersection of A with $S_\theta(p)$ is denoted by $A_\theta(p)$ and called a *section* of A. An *object* is a fuzzy subset A of S such that any $\mu_A(p)$ belongs to the set $\{\alpha_1,\alpha_2,\ldots,\alpha_{k+1}\}$, with $1=\alpha_1>\alpha_2>\ldots>\alpha_{k+1}=0$, and any $(A_\theta(p))^{\alpha_i}$ has a finite number of connected components.

Consider two objects A and B. Consider a function $\mathcal{F}$ that accepts argument values of the form $(\theta,A_\theta(p),B_\theta(p))$. The *$\mathcal{F}$-histogram* associated with the pair (A,B) is a function $\mathcal{F}^{AB}$ of θ. Its intended purpose is to represent, in some way, the position of A with respect to B. The histogram value $\mathcal{F}^{AB}(\theta)$ is defined as a combination of the $\mathcal{F}(\theta,A_\theta(p),B_\theta(p))$ values, for all p in $\theta^{\perp}$. See (9) and Fig. 3, where ope stands for the combination operator. Figure 4 is related, but will be commented in Section 4.

$$\mathcal{F}^{AB}(\theta) = \mathrm{ope}_{p\in\theta^{\perp}}\ \mathcal{F}(\theta,A_\theta(p),B_\theta(p)) \tag{9}$$

Typically, $\mathcal{F}$ and $\mathcal{F}^{AB}$ are real functions, the combination operator ope is the addition, and

$$\mathcal{F}^{AB}(\theta) = \int_{p\in\theta^{\perp}} \mathcal{F}(\theta,A_\theta(p),B_\theta(p))\ \mathrm{d}p\ . \tag{10}$$

The key point, then, is how to choose $\mathcal{F}$. First, we might want to reduce the handling of fuzzy sections I and J to that of crisp sections, through some other function F:

$$\mathcal{F}(\theta,I,J) = \sum_{i=1}^{k}\sum_{j=1}^{k}(\alpha_i-\alpha_{i+1})(\alpha_j-\alpha_{j+1})\,F(\theta, I^{\alpha_i}, J^{\alpha_j})\ . \tag{11}$$

Second, we might want to reduce the handling of crisp sections I and J to that of their connected components $I_1, I_2, \ldots, I_m$ and $J_1, J_2, \ldots, J_n$:

$$F(\theta,I,J) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(\theta,I_i,J_j)\ . \tag{12}$$

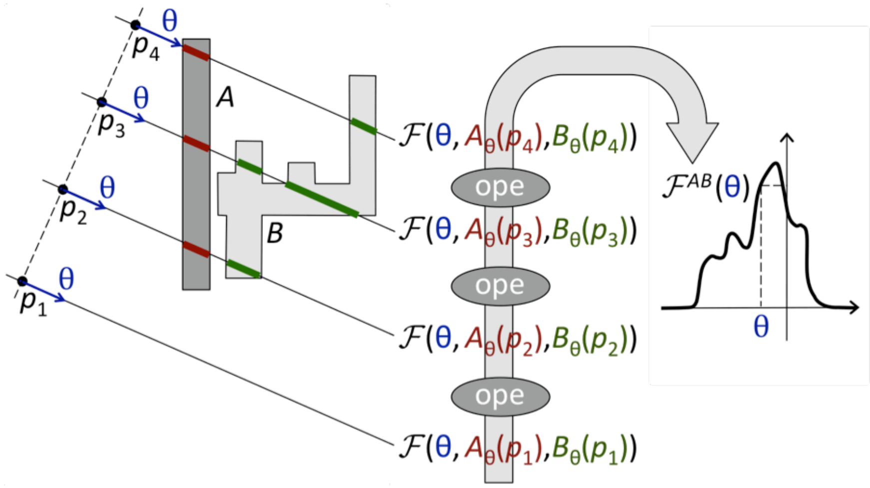


Fig. 3 Principle of the calculation of the $\mathcal{F}$-histogram $\mathcal{F}^{AB}$

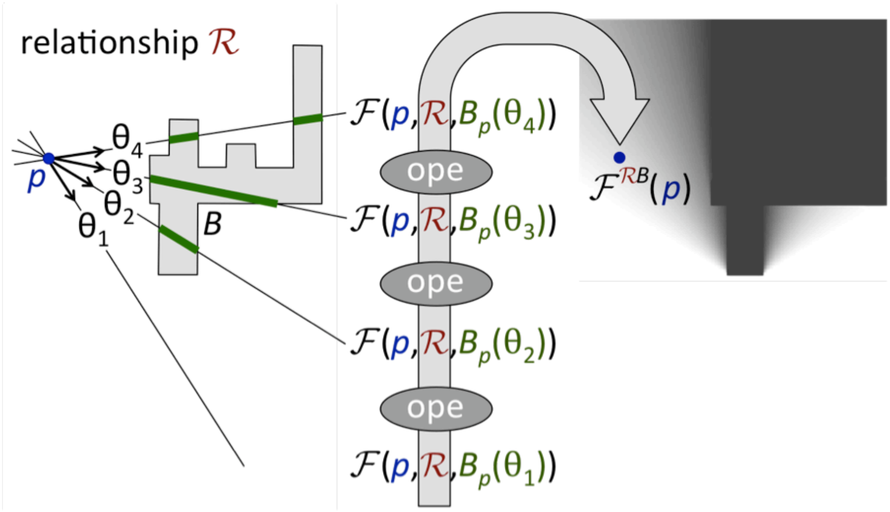


Fig. 4 Principle of the calculation of the $\mathcal{F}$-template $\mathcal{F}^{\mathcal{R}B}$

Further reduction can be expressed as

$$f(\theta, I, J) = \int_{p \in I} \int_{q \in J} \varphi(\theta, p, q)\, \mathrm{d}p\, \mathrm{d}q\ , \tag{13}$$

where I and J are (crisp) singletons, segments, lines or half-lines.

Note that for any fuzzy sections I and J, we then have

$$\mathcal{F}(\theta,I,J)=\int_{p\in S}\int_{q\in S}\mu_I(p)\,\mu_J(q)\,\varphi(\theta,p,q)\,\mathrm{d}p\,\mathrm{d}q\ . \tag{14}$$

$\mathcal{F}^{AB}$ can also be referred to as the F-histogram F^{AB}, the f-histogram f^{AB}, or the φ-histogram φ^{AB}, depending on whether (11), (11) and (12), or (11), (12) and (13) hold. This categorization is illustrated by Fig. 5.

Two properties are worth noticing at this point:

$$F^{AB}=\sum\nolimits_{i=1}^{k}\sum\nolimits_{j=1}^{k}(\alpha_i-\alpha_{i+1})(\alpha_j-\alpha_{j+1})\,F^{A^{\alpha_i}B^{\alpha_j}} \tag{15}$$

and

$$f^{(\bigcup_{i=1}^{m}A_i)(\bigcup_{j=1}^{n}B_j)}=\sum\nolimits_{i=1}^{m}\sum\nolimits_{j=1}^{n}f^{A_iB_j}\ , \tag{16}$$

where $A_1, A_2, \ldots, A_m$ are pairwise disjoint objects, and $B_1, B_2, \ldots, B_n$ too. Now, for any real number r, consider the function φ_r defined by: $\varphi_r(\theta,p,q)=1/|qp|^r$ if $p\neq q$ and if θ is the direction of the vector qp; $\varphi_r(\theta,p,q)=0$ otherwise. The φ_r-histogram φ_r^{AB} is called a *force histogram*. The reason for the term *force* (and for the symbols $\mathcal{F}$, F, f and φ, which all refer to the first letter of the words f*unction* and f*orce*) is the following. For any direction θ, the value $\varphi_r^{AB}(\theta)$ can be seen as the scalar resultant of elementary physical forces. These forces (which are additive vector quantities) are exerted by the points of A on those of B, and each tends to move B in direction θ. Assume r=2. The forces then correspond to gravitational forces. This is according to Newton's law of gravity, which states that every particle attracts every other particle with a force inversely proportional to the square of the distance between them. Under the above assumption, it is as if the objects A and B had mass and density: the area (2D case) or volumetric (3D case) mass density of A at point p is $\mu_A(p)$; the density of B at q is $\mu_B(q)$. Note that in the 2D case A and B can be seen as flat metal plates of constant and negligible thickness.

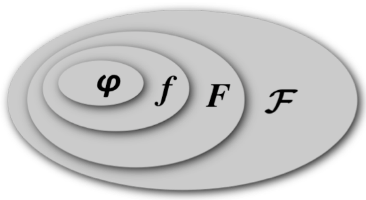


Fig. 5 Categorization of $\mathcal{F}$-histograms
$\mathcal{F}$-histograms include F-histograms, which include f-histograms, etc.

4 $\mathcal{F}$-templates

How to identify, in a scene, the object that best satisfies a given relationship to a reference object? This question, which is one of the two fundamental questions that arise when dealing with spatial relationships (Section 1), defines an object localization task. One theory supported by cognitive experiments is that people accomplish this task by parsing space around the reference object into *good* regions (where the object being sought is more likely to be), *acceptable*, and *unacceptable* regions (where the object being sought cannot be) [15] [22]. These regions blend into one another and form a so-called *spatial template* [22], which assigns each point in space a value between 0 (unacceptable region) and 1 (good region). In other words, a spatial template is a fuzzy subset of the Euclidean space that represents a relationship to a reference object. The concept of the $\mathcal{F}$-template was introduced in a series of three conference papers [29] [52] [10]. The idea and assumption behind it are that acceptable representations of relationships to reference objects can be obtained by reducing the handling of all 3D and 2D objects to the handling of 1D entities. A formal definition of the $\mathcal{F}$-template is given in Section 4.1, and it is followed by an important example in Section 4.2.

4.1 Definition

Here, the line in direction θ that passes through the point p is denoted by $S_p(\theta)$ (instead of $S_\theta(p)$, as in Section 3). The (fuzzy) intersection of $S_p(\theta)$ with a fuzzy subset A of S is denoted by $A_p(\theta)$ (instead of $A_\theta(p)$). Consider a spatial relationship $\mathcal{R}$ and an object B. Consider a function $\mathcal{F}$ that accepts argument values of the form $(p,\mathcal{R},B_p(\theta))$. The *$\mathcal{F}$-template* associated with the pair $(\mathcal{R},B)$ is a function $\mathcal{F}^{\mathcal{R}B}$ of p. Its intended purpose is to represent, in some way, the relationship $\mathcal{R}$ to the reference object B. The template value $\mathcal{F}^{\mathcal{R}B}(p)$ is defined as a combination of the $\mathcal{F}(p,\mathcal{R},B_p(\theta))$ values, for all θ. See (17) and Fig. 4, where ope stands for the combination operator.

$$\mathcal{F}^{\mathcal{R}B}(p) = \mathrm{ope}_\theta\, \mathcal{F}(p,\mathcal{R},B_p(\theta)) \tag{17}$$

There is obviously a duality between the $\mathcal{F}$-template and the $\mathcal{F}$-histogram, and it echoes the duality between the two fundamental questions mentioned in Section 1. Compare Fig. 4 with Fig. 3, and Fig. 2 with Fig. 1. Compare (17) with (9). In (17), θ varies and p does not. In (9), p varies and θ does not. Replace any subset of A with $\mathcal{R}$, replace p with θ and θ with p, and (9) transforms into (17). Typically, $\mathcal{F}$ and $\mathcal{F}^{\mathcal{R}B}$ are real functions with output values in the range [0,1], the template $\mathcal{F}^{\mathcal{R}B}$ is a fuzzy subset of the Euclidean space, and $\mathcal{F}^{\mathcal{R}B}(p)$ aims to represent the degree to which p satisfies the relationship $\mathcal{R}$ to the reference object B.

4.2 An important example: Basic directional templates

A spatial template that represents a directional relationship to a reference object may be called a *directional (spatial) template*. To emphasize the analogy with the well-known distance maps (mentioned in Section 2.2), we may also call it a *directional map*. In [2], Bloch introduces the concept of the *fuzzy landscape*. A fuzzy landscape is a specific example of directional template, which does not sacrifice the geometry of the reference object (the object is not approximated through, e.g., its centroid, or its minimum bounding rectangle or box). Moreover, the defining equation (whose roots can be traced to earlier works [34] [20]) is very simple and intuitive. Because of this and the fact that the term *template* was coined earlier, and also to increase precision in language, we prefer to talk of *basic directional templates*, or *basic directional maps*, instead of fuzzy landscapes. The basic directional template induced by the object B in direction δ associates the value

$$\sup_{q\in S-\{p\}}\ \mu_B(q)\ \min\left\{1,\max\left\{0,1-\frac{2}{\pi}\angle(qp,\delta)\right\}\right\} \tag{18}$$

with each point p, where $\angle(qp,\delta)$ denotes the angle between the two vectors qp and δ. Compare (18) with (1). In the case of raster data, the exact algorithm that naturally results from (18) is straightforward but computationally expensive. Reference [2] describes a much faster approximation algorithm, inspired by chamfer methods. Consider, e.g., a 2D image. The pixels are examined sequentially, from top to bottom and left to right, and then from bottom to top and right to left. Each time a pixel is examined, it is assigned a value whose calculation also involves the pixel's neighbors. As shown in [2], a basic directional template can be seen as the morphological dilation of the reference object by a fuzzy structuring element. The idea behind the algorithm is to perform the fuzzy dilation with a limited support for the structuring element. According to Bloch, "most approaches [e.g., the $\mathcal{F}$-histogram / template approach] reduce the representation of objects to points, segments or projections" [3] while hers "takes morphological information about the shapes… into account" [2], "considers the objects as a whole and therefore better accounts for their shape" [3]. The argument does not hold water, since basic directional templates can be proved to be $\mathcal{F}$-templates [29]. As a result, they can be calculated by reducing objects to segments, using an $\mathcal{F}$-template approach [29] [52]. An extensive experiment [30] has shown that this approach should be preferred to the morphological one in the case of 2D raster data, but that the opposite holds in the case of 3D raster data. 2D vector data can only be handled using the $\mathcal{F}$-template approach, and there is yet no algorithm for 3D vector data. Once the basic directional template induced by B in direction δ has been computed, the degree of truth of the statement "A is in direction δ of B" (i.e., "A is in relationship $\mathcal{R}$ with B" where $\mathcal{R}$ denotes the relationship "in direction δ") can be calculated for any object A, in comparatively no time, using a fuzzy pattern-matching approach [12] [2].

5 $\mathcal{F}$-histograms from spatial correlation

Most work on $\mathcal{F}$-histograms has focused on force histograms. The reasons are multiple, as explained in Section 5.1. Force histograms can actually be generated from spatial correlations; this is an important concept, covered in Section 5.2.

5.1 Interest in force histograms

Force histograms are relative position descriptors with high discriminative power [25]. Moreover, the way they change when the objects are affine-transformed is known [25] [36]. This is an important issue in computer vision and pattern recognition, especially because it is intrinsically linked to the design of widely used affine invariant descriptors. Remember that affine transformations include, e.g., translations, rotations, scalings, reflections and stretches. Let *aff* be any invertible affine transformation. It can be written as the composition of a translation with a linear transformation *lin* (an affine transformation such that *lin*(ω)=ω). It is a common convention to see *lin* as a matrix (a 2×2 matrix if *S* is of dimension 2; a 3×3 matrix if *S* is of dimension 3). Likewise, vectors can be seen as column matrices and vice versa. As shown in [36], for any real number *r*, any objects *A* and *B*, and any direction θ,

$$\varphi_r^{aff[A]\,aff[B]}(\theta) = \left|\det(lin)\right| \left|lin^{-1} \cdot \theta\right|^{r-1} \varphi_r^{AB}(\theta')\,. \tag{19}$$

In this equation, det(*lin*) is the determinant of the matrix *lin* and $|\det(lin)|$ its absolute value; the symbol · denotes matrix multiplication; $|lin^{-1} \cdot \theta|$ is the norm of the vector $lin^{-1} \cdot \theta$; the direction θ' is the unit vector $(lin^{-1} \cdot \theta) / |lin^{-1} \cdot \theta|$. The importance of having a property such as (19) is discussed in [25] and illustrated through experiments with synthetic and real data.

Another reason for the special interest in force histograms is that they lend themselves, with great flexibility, to the modeling of directional relationships by fuzzy binary relations [26]. The main methods that can be used to achieve this are the aggregation method [20], the compatibility method [34], and the method based on force categorization [27]. The fuzzy relations then satisfy four fundamental properties, which express the following intuitive ideas: if the objects in hand are sufficiently far apart, each one can be seen as a single point in space; the directional relationships are not sensitive to scale changes; all directions have the same importance; the semantic inverse principle [16] is respected (e.g., object *A* is to the left of object *B* as much as *B* is to the right of *A*). As a corollary of these properties, it is possible to determine how the fuzzy relations react when the objects are similarity-transformed. There is, of course, a link with (19), since similarity transformations are particular affine transformations. Note that the four abovementioned properties form the axiomatic basis upon which the concept of the histogram of forces was actually developed [23].

Directional relationships are not, however, the only spatial relationships that can be assessed from force histograms. Reference [42] describes a fuzzy model of inner-adjacency. The position of A relative to B is then represented by $\varphi_r^{A(B-A)}$ instead of φ_r^{AB}. Reference [45] describes a fuzzy model of surroundedness. The underlying assumption is that A is connected and does not intersect the convex hull of B. Reference [24] describes a fuzzy model of betweenness. Although the preposition "between" usually denotes a ternary relationship, its model in [24] is a fuzzy binary relation. A sentence such as "A is between B and C" is read "A is between $B \cup C$". The position of A relative to B and C is represented by $\varphi_r^{A(B \cup C)}$.

Most work on $\mathcal{F}$-histograms has focused on force histograms, but not all. Consider φ_2^{AB}. It has interesting characteristics [33]. Usually, however, it is not defined anywhere if A and B intersect, because the integral in (10) then diverges. As shown in [23], φ-histograms that are not force histograms make it possible to overcome this limitation while preserving the abovementioned characteristics. Reference [23] also suggests f-(non-φ-)histograms for the handling of convex objects. The fuzzy model of surroundedness mentioned in the previous paragraph suits the application considered in [45], but only because the objects there satisfy certain conditions. Another model would otherwise be necessary. Its design could be based on F-(non-f-)histograms, instead of force histograms. This is a promising avenue, as pointed in [24]. Finally, [31] describes $\mathcal{F}$-(non-F-)histograms for the combined extraction of directional and topological relationship information. The particularity of these histograms is that they are coupled with Allen relations [1] using fuzzy set theory. Various systems rely on them to capture the essence of the relative positions of objects with natural language descriptions [32] [55] [56].

5.2 Spatial correlation

In [27], a natural language description of the relative position between two objects A and B is generated from the force histograms φ_0^{AB} and φ_2^{AB}. The fact is that φ_0^{AB} and φ_2^{AB} have very different and interesting characteristics, which complement one another. As this example shows, it may be useful to calculate two or more force histograms associated with the same pair of objects. These histograms are obviously not totally independent from each other. Part of the calculation of one might therefore be common to all and yield an intermediate result, interesting if only for efficiency purposes. The same idea was expressed in Section 1; the intermediate result was seen as a quantitative representation of the relative position between two objects. Here, the intermediate result is a *spatial correlation*. Compare Fig. 6 with Fig. 1. Figure 7 is related, but will be commented in Section 6. The spatial correlation between A and B provides raw information about the position of A relative to B. It is the function ψ^{AB} defined by

$$\psi^{AB}(v) = \int_{q \in S} \mu_A(q+v)\,\mu_B(q)\,\mathrm{d}q\ , \tag{20}$$

where v denotes any vector and + denotes point-vector addition. All force histograms φ_r^{AB} associated with A and B can be derived from ψ^{AB} as follows:

$$\varphi_r^{AB}(\theta) = \int_0^{+\infty} \frac{\psi^{AB}(u\theta)}{u^r}\,\mathrm{d}u\ . \tag{21}$$

Reference [36] shows that (20) and (21) lead to different algorithms than (10) and (14) and are better adapted to the solving of some theoretical issues.

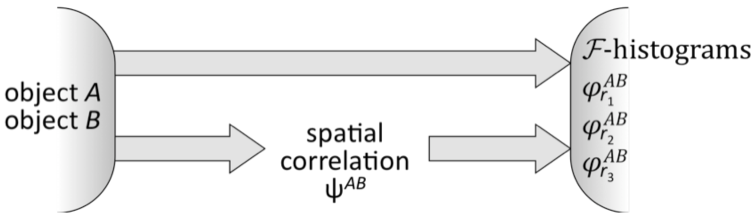


Fig. 6 $\mathcal{F}$-histograms from spatial correlation

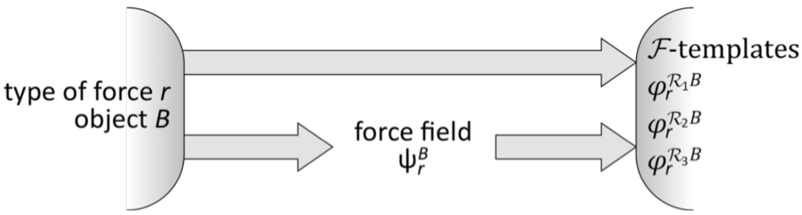


Fig. 7 $\mathcal{F}$-templates from force field

6 $\mathcal{F}$-templates from force field

Basic directional templates have been used for spatial reasoning, object localization and identification, structural and model-based pattern recognition [4] [9] [21] [48]. They have, however, important flaws. They are overly sensitive to outliers. Elongated reference objects pose problems, and concave objects as well [28]. The main reason is that basic directional templates make use of angle information but ignore distance information. According to cognitive experiments [22] [17] [15], the former is indeed of primary importance, but the latter also contributes in shaping a directional template. For a given angular deviation, the membership degrees are not constant. They fluctuate slightly, depending on the distance to the reference object. Moreover, the fluctuation varies from one angular deviation to another. Finally, when sufficiently far from the object, all the membership degrees drop. For example, if you were told that the soccer ball was to the right of the bench, you would not look for it hundreds of feet from the bench.

One may wonder whether angle information and distance information could be processed in separate steps. In [52], the authors argue that the answer is negative, and they show how directional $\mathcal{F}$-templates can embed distance information to elegantly overcome the abovementioned flaws. Their work is based on the following results: *(i)* basic directional templates are $\mathcal{F}$-templates [29]; *(ii)* distance maps like d_B and $\overline{d}_B$ (Section 2.2) are $\mathcal{F}$-templates too [52]; a binary operation $\otimes$ and two $\mathcal{F}$-templates $p \mapsto \text{ope}_\theta\, \mathcal{F}_1(p,\mathcal{R},B_p(\theta))$ and $p \mapsto \text{ope}_\theta\, \mathcal{F}_2(p,\mathcal{R},B_p(\theta))$ define a new $\mathcal{F}$-template $p \mapsto \text{ope}_\theta\, \mathcal{F}_1(p,\mathcal{R},B_p(\theta)) \otimes \mathcal{F}_2(p,\mathcal{R},B_p(\theta))$.

Now, assume different directional relationships to the same reference object need to be considered. Assume they are represented by directional templates. These templates are obviously not totally independent from each other. Part of the calculation of one might therefore be common to all and yield an intermediate result, interesting if only for efficiency purposes. The same idea was expressed in Section 1; the intermediate result was seen as a quantitative representation of a relationship to a reference object. Here, the idea is coupled with the desire to exploit the duality between $\mathcal{F}$-templates and $\mathcal{F}$-histograms; the intermediate result is a *force field*. Compare Fig. 7 with Fig. 2, and Fig. 7 with Fig. 6.

Once the force field has been computed, $\mathcal{F}$-templates that represent directional relationships to the reference object can be derived from the field in negligible time. Basic directional templates, and the templates described in [52], cannot be calculated using such a two-step procedure. The force field induced by B is the function ψ_r^B defined as follows:

$$\psi_r^B(p) = \int_{q \in S} \mu_B(q) \frac{qp}{|qp|^{r+1}} \mathrm{d}q \tag{22}$$

The reason for the term *force* is the same as in Section 3. The object B is seen as an object with mass and density: the density of B at point q is $\mu_B(q)$. The vector $\psi_r^B(p)$ is the force exerted on B by a particle of mass 1 located at p. The force field-based template induced by B in direction δ makes use of both angle and distance information. It is a function $\varphi_r^{\mathcal{R}B}$ that may be defined as

$$\varphi_r^{\mathcal{R}B}(p) = \max\left\{0, \frac{\psi_r^B(p) \bullet \delta}{\sup_{q \in S} \psi_r^B(q) \bullet \delta}\right\}, \tag{23}$$

where $\bullet$ denotes the dot product and $\mathcal{R}$ is the relationship "in direction δ". Once $\varphi_r^{\mathcal{R}B}$ has been computed, the degree of truth of the statement "A is in relationship $\mathcal{R}$ with B" (i.e., "A is in direction δ of B") can be calculated for any object A, the same way as mentioned in Section 4.2. Preliminary experiments [28] [38], where the characteristics of force field-based templates are examined and compared with those of basic directional templates, show the interest of the approach. Note that the connection between the two pairs $(\psi^{AB}, \varphi_r^{AB})$ and $(\psi_r^{B}, \varphi_r^{\mathcal{R}B})$ in Figs. 6 and 7 can be elegantly expressed by the equation below:

$$\int_\theta \varphi_{r-1}^{AB}(\theta)\ \theta\ \mathrm{d}\theta = \int_p \mu_A(p)\, \psi_r^B(p)\, \mathrm{d}p \tag{24}$$

7 On the design of efficient algorithms

$\mathcal{F}$-histograms and $\mathcal{F}$-templates lend themselves to the design of efficient algorithms, whether the Euclidean space is of dimension two or three, the objects are crisp or fuzzy, in raster or vector form. Section 7.1 illustrates some of the typical steps in the design process. These steps are briefly described in Section 7.2, from a higher perspective. An important issue (the selection of a set of reference directions) is covered more extensively in Section 7.3.

7.1 Illustrative example

How can (10) and (14) be adapted to the case of 2D raster data? Consider two objects A and B, a direction θ and a point p. As illustrated in Fig. 8a, the line $S_\theta(p)$ might pass through some pixels i and j of A and B, with nonzero membership degrees $\mu_A(i)$ and $\mu_B(j)$. These pixels can be determined by rasterizing $S_\theta(p)$ using a line-drawing algorithm. They project on $S_\theta(p)$ as segments I_i and J_j. Let I=$A_\theta(p)$ and J=$B_\theta(p)$. The value of $\mathcal{F}(\theta,I,J)$ may be calculated as follows:

$$\mathcal{F}(\theta,I,J) = \sum_i \sum_j\ \mu_A(i)\, \mu_B(j) \int_{p\in I_i} \int_{q\in J_j} \varphi(\theta,p,q)\, \mathrm{d}p\, \mathrm{d}q \tag{25}$$

(25) then replaces (14). Moreover, the integral in (10) can be approximated by a finite sum; (10) may be replaced with

$$\mathcal{F}^{AB}(\theta) = \varepsilon_\theta \sum_{k\in\mathbb{Z}} \mathcal{F}(\theta, A_\theta(p_k), B_\theta(p_k))\ , \tag{26}$$

where ε_θ and the points p_k are as suggested in Fig. 9a. Symbolic computation of the double integral in (25) yields closed-form expressions that do not depend on A nor B. See Fig. 10. In the end, numerical computation of $\mathcal{F}(\theta,I,J)$ — and $\mathcal{F}^{AB}(\theta)$ — translates into multiple instantiations of these expressions.

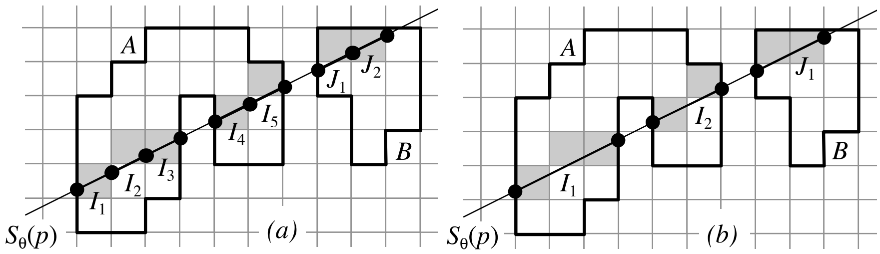


Fig. 8 The sections $A_\theta(p)$ and $B_\theta(p)$ are decomposed into segments I_i and J_j. In the case of fuzzy objects (left), all segments are of the same length. In the case of crisp objects (right), the segments I_i are mutually disjoint, and the segments J_j also.

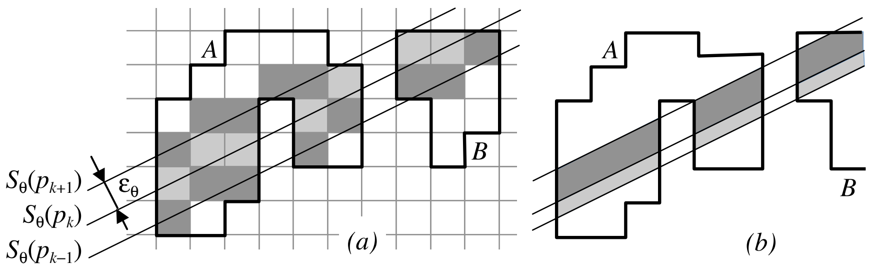
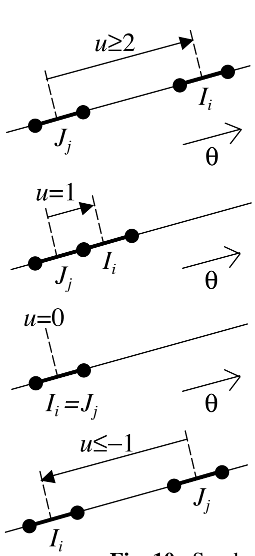


Fig. 9 In the case of raster data (left), the lines $S_\theta(p_k)$ partition the objects into sets of adjacent pixels; the distance between two consecutive lines is constant. In the case of vector data (right), the lines pass through the vertices of the objects and partition the objects into trapezoids; the distance between two consecutive lines varies.



Case		$\int_{p\in I_i}\int_{q\in J_j} \varphi_r(\theta,p,q)\,\mathrm{d}p\,\mathrm{d}q$
$u \geq 2$	$r=1$	$\ell\,[(u-1).\ln(u-1)-2u.\ln(u)+(u+1).\ln(u+1)]$
	$r=2$	$2.\ln(u)-\ln(u-1)-\ln(u+1)$
	$r\neq 1$ and $r\neq 2$	$\ell^{2-r}\,[(u-1)^{2-r}-2u^{2-r}+(u+1)^{2-r}]\ /\ [(1-r)(2-r)]$
$u=1$	$r=1$	$2\ \ell\ \ln(2)$
	$r\neq 1$ and $r<2$	$\ell^{2-r}\,(2^{2-r}-2)\ /\ [(1-r)(2-r)]$
	$r\geq 2$	$+\infty$
$u=0$	$r<1$	$\ell^{2-r}\ /\ [(1-r)(2-r)]$
	$r\geq 1$	$+\infty$
$u\leq -1$		0

Fig. 10 Symbolic integration. Several cases must be considered, depending on r and on the position of I_i relative to J_j. These segments are of length ℓ.

For crisp objects, (25) can be rewritten as follows:

$$\mathcal{F}(\theta,I,J)=\sum_i\sum_j \int_{p\in I_i}\int_{q\in J_j} \varphi(\theta,p,q)\,\mathrm{d}p\,\mathrm{d}q\,, \tag{27}$$

where the segments I_i and J_j are now as illustrated in Fig. 8b. In this case, symbolic computation of the double integral yields more expressions than as in Fig. 10. $\mathcal{F}^{AB}(\theta)$, however, computes much faster, since each instantiation corresponds to the process of a batch of pairs of object pixels instead of the process of a single pair. Actually, (27) can be used in place of (25) whether the objects are crisp or fuzzy: the idea is to exploit (15), i.e., it is to handle the fuzzy objects through their level cuts (which are crisp objects). Equations (15) and (27) lead to shorter processing times than (25) if one object is crisp and the other fuzzy with few membership degrees. Note that all of the above holds in the case of 3D raster data: replace the word 'pixel' with 'voxel', and the sum in (26) with a double sum. Vector data require more important changes: these data can be handled very efficiently, because the objects can be partitioned into bigger blocks (Fig. 9b); however, the symbolic integration step is more complex and generates a higher number of closed-form expressions.

7.2 Typical steps

Typical steps in the design of efficient algorithms include the following. First, a set of reference directions is chosen (more on this in the next section). Then, for every reference direction θ, a partitioning of the space is undertaken. When dealing with vector data, each block of the partition is a region of space delimited by two lines in direction θ (2D case), or by planes that include such lines (3D case). When dealing with raster data, each block corresponds to a raster line in direction θ, or to a region of the image defined by the union of such lines. The blocks, in turn, cut the considered object(s) into pieces, and vice versa. The whole partitioning procedure can rely on efficient computer graphics software and hardware tools. For example, line-drawing algorithms such as Bresenham's [6] are often implemented in the firmware or hardware of graphics cards, and graphics accelerators provide operations such as polygon clipping implemented in high-speed hardware. Once the partitioning procedure is completed, the different blocks can be processed independently from each other. The same, of course, applies to the reference directions. The algorithms for the computation of $\mathcal{F}$-histograms and $\mathcal{F}$-templates are, therefore, highly parallelizable. When equations like, e.g., (13) or (22) are involved, an additional and more common way to increase efficiency is to harness the power of integral calculus. A definite integral can be approximated by a finite sum. One may use, e.g., a Riemann sum, the trapezoidal rule, Simpson's rule, or any other Newton-Cotes rule. Different algorithms may actually result from this procedure, depending, e.g., on whether the integral is written in Cartesian or polar coordinates. In some cases, however, symbolic integration can be performed and closed formulas obtained. The domain of integration and the integrand usually involve various parameters. One single integral may therefore correspond to one, a few, tens, or even hundreds of formulas, which can be hard-coded and organized in a tree structure. The appropriate formula can then be found and instantiated at run-time, when the values of the parameters are known. The procedure is particularly efficient. Note that crisp and vector data tend to lend themselves to symbolic integration more easily than fuzzy or raster data.

7.3 Set of reference directions

An important issue is how to choose the set of reference directions. Practically, of course, only a finite number K of directions can be considered. The higher K, the more complete the collected $\mathcal{F}$-histogram data, or the more accurate the computed $\mathcal{F}$-template values, but the longer the processing time. Usually, the reference directions are so chosen as to satisfy the following properties: they are evenly distributed in space; they include the primitive directions (right and left, above and below, front and behind); if θ is a reference direction, its opposite $-\theta$ (which can be processed at the same time) also is a reference direction.

In the 2D case, the standard procedure is therefore to pick the directions defined by the angles $2\pi i/K$, for all i in $0..K-1$, with K a multiple of 4. The value for K can be as low as 4, and arbitrarily high, but a few tens to a few hundreds of reference directions, with a maximum of 360, seem to be necessary and sufficient for most applications. Note that when the objects are in raster form, K is naturally limited by the size of the image. In an $n\times n$ image, for example, the largest set of directions worth considering is a set of $8n-8$ unevenly distributed directions [35] [36].

In the 3D case, finding an arbitrarily large set of evenly distributed directions is not obvious. One might want to pick the directions $\omega p/|\omega p|$, for all vertices p of a regular convex polyhedron centered at ω. However, there exist only 5 regular convex polyhedra (the Platonic solids), and none has more than 20 vertices. A solution is to calculate (e.g., using a random-start hill-climbing heuristic) the equilibrium positions, on a sphere centered at ω, of K points p that repulse each other like equally charged particles [37]. Obviously, to get the same density of information, many more reference directions are needed in 3D than in 2D. For example, 30 directions in 2D correspond roughly to 300 directions in 3D (neighbor particles on the unit sphere will then be about $2\pi/30$ apart). 300 directions in 2D correspond to 30,000 in 3D. If the running time of the algorithm is linear in K, which is typical, one should definitely consider parallel computing. Some algorithms, however, focus on the calculation of intermediate data, and their running time is practically independent of K [35] [36]. Also note that K can be dynamically increased: since directions are handled independently from each other, additional ones can be considered in subsequent stages, depending on results and time constraints.

A peculiar situation is worth mentioning: it may happen that $\mathcal{F}^{AB}(\theta)$=0 for all reference directions θ (Fig. 11a). One may then force $\mathcal{F}^{AB}(\theta_0)$ to $\mathcal{F}^{AB}(qp/|qp|)$, where p is an arbitrary point of A and q is an arbitrary point of B, and where θ_0 is the reference direction closest to $qp/|qp|$ (i.e., the dot product of the two vectors is maximum). In this situation, however, the objects might be too far apart, and a simpler model than the $\mathcal{F}$-histogram might be sufficient (e.g., a model based on minimum bounding rectangles or boxes); the objects might not be appropriate for the model (e.g., clouds of scattered, small connected components); the number of reference directions might be too low (note that this number can be easily adapted to the objects, as illustrated in Fig. 11b). Similar comments apply to $\mathcal{F}$-templates (Fig. 11c).

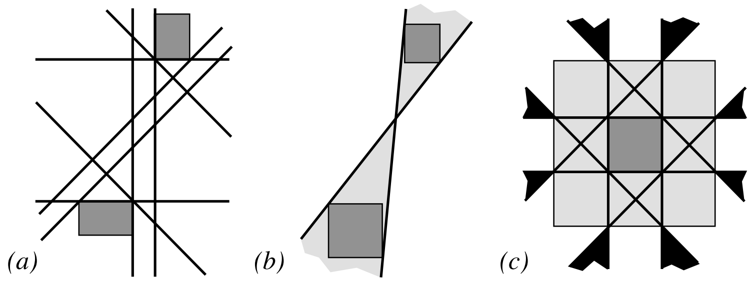


Fig. 11 Each dark gray rectangle represents (the minimum bounding rectangle of) some 2D object. The reference directions are the horizontal, vertical and diagonal directions. *(a)* All computed $\mathcal{F}$-histogram values are 0. *(b)* The number of reference directions may be chosen depending on the angle between the two lines. *(c)* The $\mathcal{F}$-template values in the black areas must be calculated independently from the others. These areas are outside the region of interest (light gray rectangle) defined by the reference directions. The higher the number of directions, the larger the region of interest.

8 Applications and related literature

Numerous applications of the general approach described in this paper have been studied, and new applications continue to be explored. Here is a review of the related literature.

Relative position descriptors like force histograms are orthogonal—and, therefore, constitute a natural complement—to color, texture, and shape descriptors. Reference [25] illustrates the point and explores the behavior of force histograms towards affine transformations. The findings, supported by experiments on synthetic and real data, suggest that these histograms would yield powerful edge attributes in attributed relational graphs and could be of great use in scene matching. The latter idea is examined and validated in [43]. The authors present a system able to determine whether two images acquired under different viewing conditions capture the same scene. If the answer is positive, the system produces a mapping of objects from one view to the other and recovers the viewing transformation parameters. The approach is based, of course, on the computation and geometric properties of force histograms.

The $\mathcal{F}$-histogram $\mathcal{F}^{AA}$ is called the $\mathcal{F}$-signature of the object A [23]. In [57], $\mathcal{F}$-signatures are used to classify orbits and sinuses represented by drawings of craniums from the 3rd century A.D. The results are consistent with human responses, and the approach compares favorably with standard ones based on geometric criteria and Fourier descriptors. Reference [49] focuses on $\mathcal{F}$-signatures of fuzzy objects. A region of a grayscale image is seen as a fuzzy object whose membership grades correspond to intensity values. An $\mathcal{F}$-signature of this object can, of course, be calculated. In [49], however, $\mathcal{F}$-signatures of its α-cuts are calculated instead, and then arranged in layers to form a 3D signature. The approach makes it easier to discriminate regions with similar shapes but different gray levels. It is extended to color images and validated using real data. Note that $\mathcal{F}$-signatures of 2D objects can be easily expressed as periodic functions. For storage efficiency and noise reduction purposes, they can therefore be approximated based on the calculation of their Fourier descriptors. This is the approach adopted in [50], which deals with the recognition of graphical symbols in technical line drawings. The methodology is demonstrated using architectural drawings. Reference [41] takes advantage of another characteristic of $\mathcal{F}$-signatures: they do not require objects to be connected. The paper presents a geospatial information retrieval and indexing system. It brings a diverse set of technologies together with an aim of allowing image analysts to more rapidly identify relevant imagery. In particular, the system is able to retrieve database satellite images with man-made objects in specific spatial configurations. This ability comes from the fact that the several objects in a given configuration form a single disconnected object, an $\mathcal{F}$-signature of which can be calculated.

In [27], degrees of truth calculated from force histograms provide inputs to a fuzzy rule base that produces intuitive, human-like linguistic descriptions of relative positions. The system is tested on regions from laser radar range images of a complex power plant scene. The same system is used in [45], where a mobile robot describes its environment based on readings from a ring of sonar sensors. Experiments are carried out with the Nomad simulator. Note that, in [27], the force histograms are computed from raster data, the spatial reference frame is implicitly determined by the reader's location (world view), and the linguistic descriptions involve directional relationships only. In [45], however, the histograms are computed from vector data, the reference frame is determined by the intrinsic orientation of the robot (egocentric view), and surroundedness is also considered. As a further step, the system is coupled with a multimodal robot interface [46]. This time, spatial information is extracted from an evidence grid map, which is built from range sensor data accumulated over time. Real examples of natural dialogs are presented. They include both robot-to-human feedback and human-to-robot commands. Another system for generating linguistic descriptions is worth mentioning. In [32], the descriptions are generated from Allen $\mathcal{F}$-histograms [31] [54] instead of force $\mathcal{F}$-histograms, and they are built around topological relationships. The approach is validated using several sets of real and synthetic data. References [27] and [32] show the specificity and limits of each type of histogram, and they show how each one can contribute to the generation of natural language expressions that capture the essence of relative positions. Finally, let us mention [44], which deals with hand-sketched route maps. Such a map does not generally contain complete map information and is not necessarily drawn to scale, but yet it contains the correct qualitative information for route navigation. The system presented in [44] is able to generate, from the sketch, a natural language description of the route to follow. The methodology is based on the use of force histograms as relative position descriptors. It is demonstrated using example sketches drawn on a handheld PDA.

The concept of the $\mathcal{F}$-template is too recent to be the subject of application papers. At this time of writing (August 2009), [30], [28] and [38] have not even been published yet. We believe, however, that there is a real potential for the concept: first, because of the duality between the $\mathcal{F}$-template and the $\mathcal{F}$-histogram—and the fact that the latter has aroused significant interest, as illustrated above; second, because spatial templates have already been the subject of application papers [48] [4] [21] [9] [47]; third, because most of these papers make use of basic directional templates—and the $\mathcal{F}$-template approach provides new, efficient algorithms for their computation [30]; last, because force field-based templates might in many cases be preferable to basic directional templates [28] [38]. Let us briefly describe, e.g., the work in [9] and [21]. As pointed out by Logan and Sadler in [22], spatial templates can be combined to represent compound relationships to reference objects. In [9], (normalized) distance maps and basic directional templates are combined using fuzzy conjunctions and disjunctions. The template resulting from such a fusion is used to construct a new external force for a deformable model—a force that expresses constraints about spatial relationships. The approach is shown to improve the segmentation of brain subcortical structures in 3D magnetic resonance images. Reference [21] describes an image retrieval system that can handle queries involving spatial relationships. The images are represented by fuzzy attributed relational graphs: each node in the graph represents an image region; each edge represents the relationships between two regions and has an attribute whose value is a tuple of degrees of truth calculated from basic directional templates. The system is tested using synthetic and natural image databases.

9 Directions for future work

Several algorithms for the computation of force histograms in the case of 2D raster data have been implemented and are worth considering. The traditional algorithm [23] [33] runs in $\mathcal{O}(Kk^2N\sqrt{N})$ time, where K is the number of directions in which forces are considered, k is the number of possible membership degrees, and N is the number of pixels in the image. It is based on (15) and (27). A variant of the algorithm runs in $\mathcal{O}(KN\sqrt{N})$ time and is based on (25). A second variant runs in $\mathcal{O}(KkN\sqrt{N})$. A third variant, dedicated to the computation of constant force histograms ($r = 0$), runs in $\mathcal{O}(KN)$ [53]. The traditional algorithm and its variants rely on (10). A completely different algorithm, the correlation-based algorithm [35] [36], is in $\mathcal{O}(N\log N)$ and relies on (21). Which algorithm or variant performs better under which conditions is an issue discussed in [36]. From a theoretical point of view, extension to 3D raster data is straightforward. An implementation of the extended traditional algorithm is presented in [37]. The extended correlation-based algorithm, however, has not been implemented yet. This is the major missing piece for the handling of raster data.

Vector data have received much less attention. Only one algorithm has been developed so far for the computation of force histograms in the 2D case [23] [33]. The algorithm runs in $\mathcal{O}(Kk^2\,\eta\,\log\,\eta)$ time, where η is the total number of object vertices, and relies on (10) and (15). A variant runs in $\mathcal{O}(Kk\,\eta\,\log\,\eta)$. The current implementation, however, can only handle disjoint crisp objects. Extension to 3D vector data can be easily achieved. For example, partition the Euclidean space using equidistant parallel planes. Each plane P_i intersects the 3D objects A and B in the 2D objects A_i and B_i. Compute the histograms φ_r^{AiBi} using the algorithm for 2D vector data. For any direction θ in the planes, $\varphi_r^{AB}(\theta)$ can be approximated by the Riemann sum $\Sigma_i\ \varphi_r^{AiBi}(\theta)\ d$, where d is the distance between two consecutive planes. This algorithm, however, has yet to be implemented.

There are also directions, for future research, of a more theoretical nature. For example, given the $\mathcal{F}$-histogram $\mathcal{F}^{AB}$, is it possible to find all the pairs (C, D) of objects such that $\mathcal{F}^{AB} = \mathcal{F}^{CD}$? Only the beginnings of an answer are given in [25]. Given the $\mathcal{F}$-histograms $\mathcal{F}^{AB}$ and $\mathcal{F}^{BC}$, is it possible to find $\mathcal{F}^{AC}$? Another question concerns the design of affine-invariant relative position descriptors. We know that the histogram of forces reacts "well" to affine transformations, in a mathematically predictable way (Section 5.1). However, the normalization procedure described in [25] leads to a similarity-invariant relative position descriptor, not to an affine-invariant descriptor. Other normalization procedures should be developed. On a different note, it would be worth investigating new fuzzy models of "surround" and "between", based on dedicated $\mathcal{F}$-histograms. Finally, let us come back to the systems for generating linguistic descriptions (Section 8). Further mechanisms to adapt these systems to individual users should be researched. Very preliminary work is reported in [5] and [51].

We have not talked about $\mathcal{F}$-templates in this section. Writing another long list of directions for future research would be useless. There is, indeed, only one question that seems to really matter at this point, and this question summarizes it all: investigate the transfer of models, properties, algorithms, etc., from $\mathcal{F}$-histograms to $\mathcal{F}$-templates, using the duality between the two concepts.

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